A Finite Element for Form-Finding and Static Analysis of Tensegrity Structures

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This text describes a mathematical model for both form finding and static analysis of tensegrity structures. A special line element that shows constant stress for any displacement of its nodes is used to define a prestressed equilibrium configuration. The form finding and static analysis are formulated as an unconstrained nonlinear programming problem, where the objective function is the total potential energy and the displacements of the nodal points are the unknowns. A quasi-Newton method is used, which avoids the evaluation of the Hessian matrix. The source and executable computer codes of the algorithm are available for download from the website of one of the authors.

Keywords: cable, element, line, minimization, nonlinear, optimization, tensegrity.

1 Introduction

A review of the important literature related to form finding methods for tensegrity structures is given by [8]. This text concentrates on the total potential energy minimization method for both form finding and static analysis of tensegrity structures. A special line element that shows constant stress for any displacement of its nodes is used to define a prestressed equilibrium configuration. The form finding and static analysis are formulated as an unconstrained nonlinear programming problem, where the objective function is the total potential energy and the displacements of the nodal points are the unknowns. A quasi-Newton method is used, which avoids the evaluation of the Hessian matrix.

The following conventions apply unless otherwise specified or made clear by the context. A Greek letter expresses a scalar. A lower case letter represents a column vector.

2 Line element definition

Figure 1 shows the geometry of the element. The nodes are labeled 1 and 2. The nodal displacements transform the element from its initial configuration to its final configuration. The strain is assumed constant along the element and the material homogeneous and isotropic.

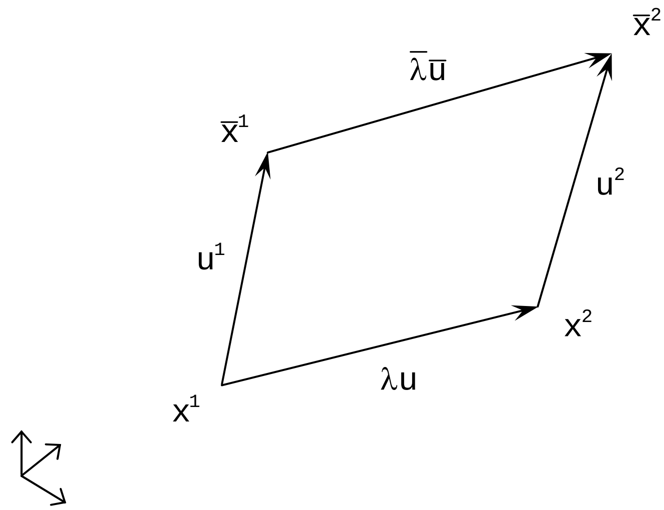

Figure 1

3 Deformed length

The vector u is a unity vector. Note that λ represents the distance between the nodes of the element in this initial configuration. However, this distance will not always represent the undeformed length of the element. The nodal displacements vectors are numbered according to its node numbers. The deformed length can be written as:

$$z = \frac{u^2 - u^1}{\lambda} \Rightarrow$$

$$\bar{\lambda}\bar{u} = \lambda u + \lambda z$$

$$\delta = 2u^T z + z^T z \Rightarrow$$

$$\bar{\lambda} = \lambda\sqrt{1 + \delta}$$

The unit vector parallel to the element in its deformed configuration can be written as:

$$\bar{u} = \frac{u + z}{\sqrt{1 + \delta}}$$

4 State of constant cut

Consider an element with undeformed length less than the initial distance of its nodes. This situation can be pictured as an imaginary cut in the element’s undeformed length. The element shows tension with zero nodal displacements. Considering μ as the value of the cut in the undeformed length, the strain-free length of the element can be written as:

$$\rho = \frac{\mu}{\lambda}$$

$$\lambda_0 = \lambda \left(1 - \rho\right)$$

The Engineering strain can be written as:

$$\varepsilon = \frac{\sqrt{1 + \delta} - 1 + \rho}{1 - \rho}$$

In order to avoid severe cancellation, the previous expression should be evaluated as:

$$\varepsilon = \frac{\dfrac{\delta}{\sqrt{1 + \delta} + 1} + \rho}{1 - \rho}$$

5 Potential strain energy

Considering σ as the conjugate stress to the Engineering strain ε and α as the undeformed area of the element, the potential strain energy and its gradient with respect to the nodal displacements can be written as:

$$\phi = \alpha\lambda_0 \int_0^{\varepsilon} \sigma\left(\xi\right) d\xi$$

$$\frac{\partial\phi}{\partial u_i^1} = \alpha\lambda_0\sigma(\varepsilon)\frac{\partial\varepsilon}{\partial u_i^1}$$

$$\frac{\partial\phi}{\partial u_i^1} = -\alpha\sigma(\varepsilon)\,\bar{u}_i$$

$$\frac{\partial\phi}{\partial u_i^2} = \alpha\lambda_0\sigma(\varepsilon)\frac{\partial\varepsilon}{\partial u_i^2}$$

$$\frac{\partial\phi}{\partial u_i^2} = +\alpha\sigma(\varepsilon)\,\bar{u}_i$$

Note that the conjugate stress to the Engineering strain is not the Cauchy stress. However, for practical purposes where the strain is usually small, this stress can be taken as an approximation to the Cauchy stress.

6 Geometric interpretation

Figure 2 shows the geometric interpretation of the gradient of the potential strain energy as forces acting on nodes of the element. These forces are known as internal forces.

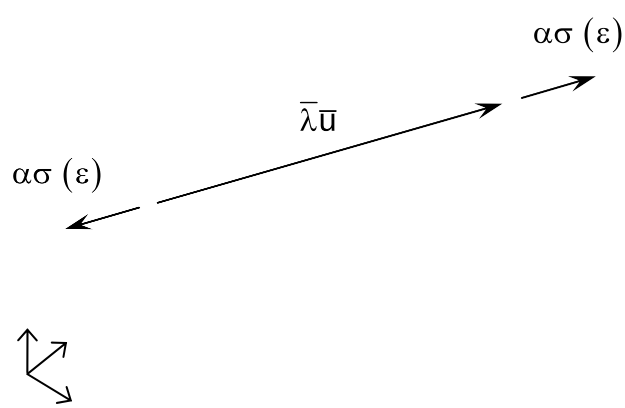

Figure 2

7 State of constant stress

Consider an element with strain-free length given by λ_0. The Engineering strain can be written as:

$$\varepsilon = \frac{\bar{\lambda} - \lambda_0}{\lambda_0}$$

Supposing that the element shows a constant stress σ, the potential strain energy can be written as:

$$\phi = \alpha\lambda_0\int_0^{\varepsilon} \sigma d\xi = \alpha\sigma\left(\bar{\lambda} - \lambda_0\right)$$

The potential strain energy is equal to the force multiplied by the relative displacement between the nodes. In the expression for the strain energy, the strain-free length can be eliminated because it does not depend on the nodal displacements. Its permanence in the expression would only add constants, one for each element, to the total potential energy function. To minimize a function plus a constant is equivalent to minimize the function only. Therefore, the potential strain energy can be defined as:

$$\phi = \alpha\sigma\bar{\lambda} = \alpha\sigma\lambda\sqrt{1 + \delta}$$

The gradient with respect to the nodal displacements can be written as:

$$\frac{\partial\phi}{\partial u_i^1} = -\alpha\sigma\bar{u}_i$$

$$\frac{\partial\phi}{\partial u_i^2} = +\alpha\sigma\bar{u}_i$$

The gradient can be interpreted as internal forces with constant modulus acting on nodes of the element. The element shows constant stress for any displacement of its nodes. A similar element was described by [4]. The element was called variable initial length element.

8 Internal forces equivalence

A cut value is equivalent to a stress value in the sense that they both produce the same internal forces. The following approach can be used when stress is a nonlinear invertible function of strain.

To find the cut value equivalent to the stress value, first find the strain according:

$$\sigma(\varepsilon) = \sigma$$

Then, find the cut value according:

$$\mu = \frac{\lambda}{(1 + \varepsilon)}\left(\varepsilon - \frac{\delta}{1 + \sqrt{1 + \delta}}\right)$$

To find the stress value equivalent to the cut value, first find the strain according:

$$\varepsilon = \frac{\frac{\delta}{\sqrt{1 + \delta} + 1} + \rho}{1 - \rho}$$

Then, find the stress according:

$$\sigma = \sigma(\varepsilon)$$

9 Stress and strain

For simplicity, consider a linear function with E as the modulus of elasticity. The potential strain energy can be written as:

$$\sigma = E\varepsilon$$

$$\phi = \frac{1}{2}\alpha\lambda_0 E\varepsilon^2$$

10 Analysis strategy

The initial configuration of a tensegrity structure is defined as the configuration of zero nodal displacements for all its nodes. An analysis strategy can be defined as: Select some elements and set them to the constant stress state by specifying a stress value. Find the prestressed equilibrium configuration. At this equilibrium configuration, change the elements from the constant stress state to its equivalent constant cut state. Note that the equilibrium configuration remains the same. Apply the loading and find the final equilibrium configuration.

11 Equilibrium configuration

Considering u as the vector of unknown displacements and f as the vector of applied nodal forces, the total potential energy function and its gradient can be written as:

$$\pi(\mathrm{u}) = \sum_{\text{elements}} \phi(\mathrm{u}) - \mathrm{f}^T \mathrm{u}$$

$$\nabla\pi(\mathrm{u}) = \sum_{\text{elements}} \nabla\phi(\mathrm{u}) - \mathrm{f}$$

The stable equilibrium configurations correspond to local minimum points of the total potential energy function. In order to find the local minimum points of a nonlinear multivariate function, the general strategy that can be used is: Choose a starting point and move in a given direction such that the function decreases. Find the minimum point in this direction and use it as a new starting point. Continue this way until a local minimum point is reached. The problem of finding the minimum points of a nonlinear multivariate function is replaced by a sequence of sub problems, each one consisting of finding the minimum of a univariate nonlinear function. In the quasi-Newton methods, starting with the unit matrix, a positive definite approximation to the inverse of the Hessian matrix is updated at each iteration. This update is made using only values of the gradient vector. A direction such that the function decreases is calculated as minus the product of this approximation of the inverse of the Hessian matrix and the gradient vector calculated at the starting point of each iteration. Consequently, it is not necessary to solve any system of equations. Moreover, the analytical derivation of an expression for the Hessian matrix is not necessary. Note that by minimizing the total potential energy function it is almost impossible to find an unstable equilibrium configuration, which corresponds to a local maximum point. The only exception is that it is possible to find a saddle point, that is, the point is a local minimum and also a local maximum. However, even in this improbable situation, a direction of negative curvature to continue toward a local minimum point can be found as described by [3]. It is important to emphasize that minimizing total potential energy to find equilibrium configurations does not require support constraints to prevent rigid body motion. The computer code uses the limited memory BFGS to tackle large scale problems as described by [6]. It also employs a line search procedure through cubic interpolation as described by [6].

12 Geometrical shape minimization

Consider the following special case: A structure composed of elements in the state of constant stress with stress equal to one and elements in the state of constant cut with cut equal to zero. The area is equal to one for all elements. The vector of applied nodal forces is equal to zero. The stress strain relationship is given by a linear function with E as the modulus of elasticity. The total potential energy can be written as:

$$\pi(u) = \sum_{stress} \lambda\sqrt{1+\delta} + \frac{E}{2}\sum_{cut} \lambda\varepsilon^2$$

The strain energy of an element in the state of constant stress is simply its final length. A high modulus of elasticity imposes resistance for changing the length of an element in the state of constant cut. The strain energy of an element in the state of constant cut can be interpreted as a penalty function. The problem can be interpreted as a constrained nonlinear programming problem of minimizing the sum of the lengths of the elements in the state of constant stress while keeping the lengths of the elements in the state of constant cut.

13 Examples

Elements shown in red are in compression. Elements shown in blue are in tension. Elements in the initial configuration that start in the state of constant stress are shown in green.

Example 1: A two element truss with a vertical downward force applied on the center node. The force was calculated to make the element rotates -45 degrees from the prestressed configuration. The analytical expression for the equilibrium equation is presented in appendix 1. The parameter values according to the definitions in this appendix are: θ_0 = 45 degrees, λ_0 = 1, E = 1000, α = 1, σ_μ = 1 and f = -587.7864. Table 1 shows the values for the axial force.

Table 1

Analytical	Numerical	Error
415.6278	415.6278	0%

Figure 3 shows the initial configuration on the left and the prestressed configuration on the right.

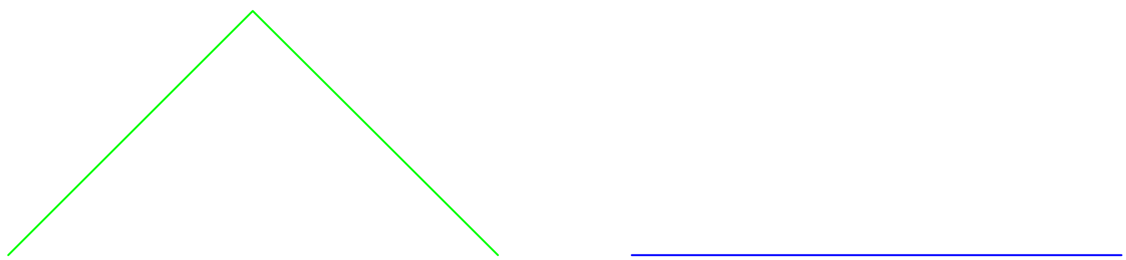

Figure 3

Figure 4 shows the prestressed configuration on the left and the loaded configuration on the right.

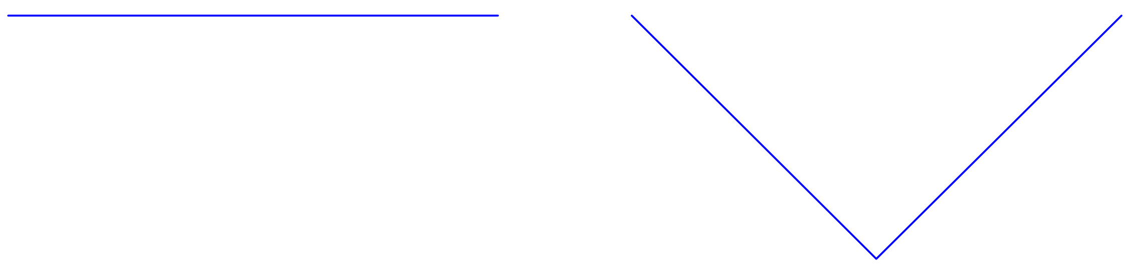

Figure 4

Example 2: A pentagonal prismoid with vertical forces applied on the top nodes. The bottom nodes are fixed only in the vertical direction. The analytical expression for the equilibrium equation is presented in appendix 2. The parameter values according to the definitions in this appendix are: n = 5, ν = 3, ρ = 1, E = 1000 for the diagonal elements, E = 1000000 to simulate the inextensible elements, α = 1 and σ_μ = 1. Figure 5 shows the initial configuration on the left and the prestressed configuration on the right. The rotation angle from the initial configuration is equal to 126 degrees.

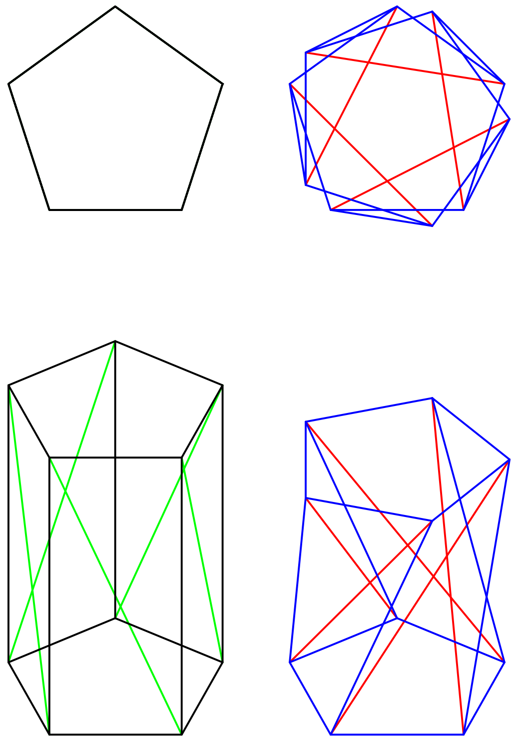

Figure 5

Figure 6 shows the prestressed configuration on the left and the loaded configuration on the right for f = 20. The rotation angle from the prestressed configuration is approximately -34 degrees.

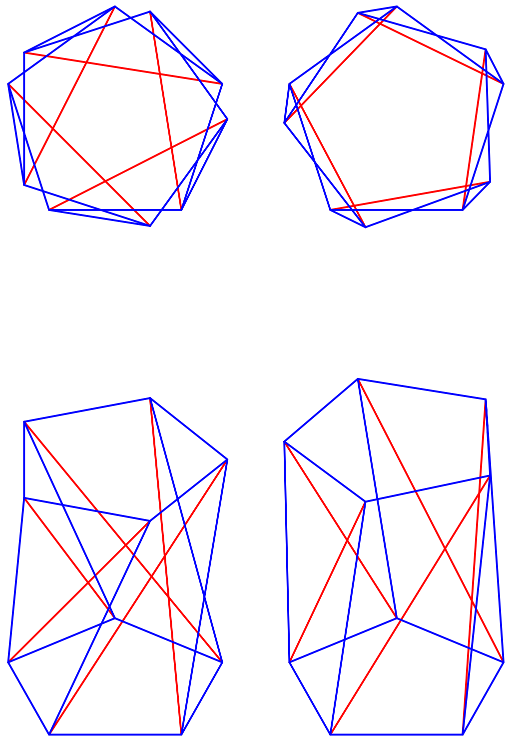

Figure 6

Figure 7 shows the prestressed configuration on the left and the loaded configuration on the right for f = -20. The rotation angle from the prestressed configuration is approximately 27 degrees.

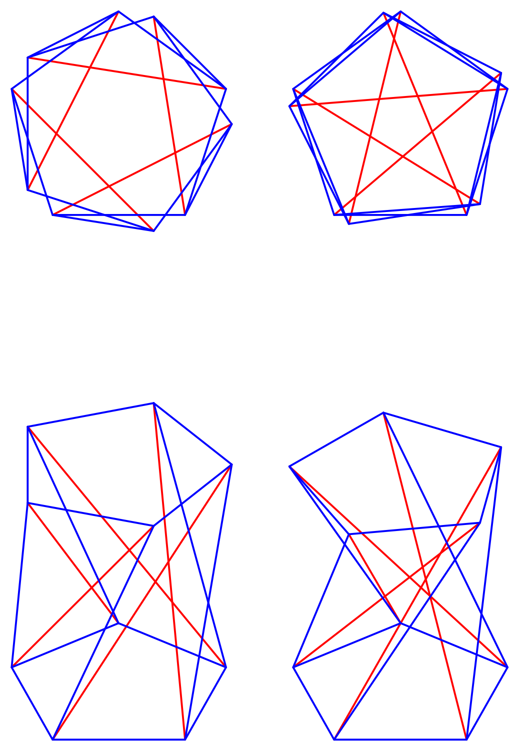

Figure 7

Table 2 shows the values for the axial force on the diagonal elements.

Table 2

f	Analytical	Numerical	Error
20.0	30.72	30.71	-0.03%
-20.0	19.83	19.80	-0.15%

Example 3: This is the same structure described in example 2, except that E = 1000 for all elements. Figure 8 shows the initial configuration on the left and the prestressed configuration on the right. The rotation angle from the initial configuration is equal to 126 degrees.

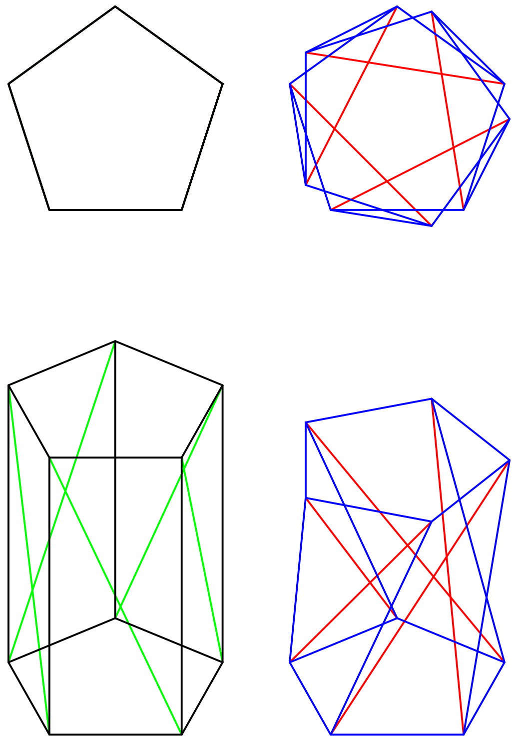

Figure 8

Figure 9 shows the prestressed configuration on the left and the loaded configuration on the right for f = 20. The rotation angle from the prestressed configuration is approximately -38 degrees.

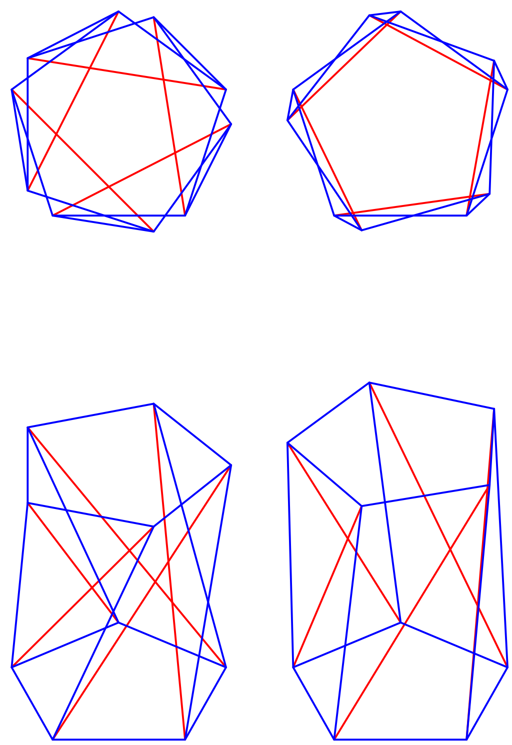

Figure 9

Figure 10 shows the prestressed configuration on the left and the loaded configuration on the right for f = -20. The rotation angle from the prestressed configuration is approximately 45 degrees.

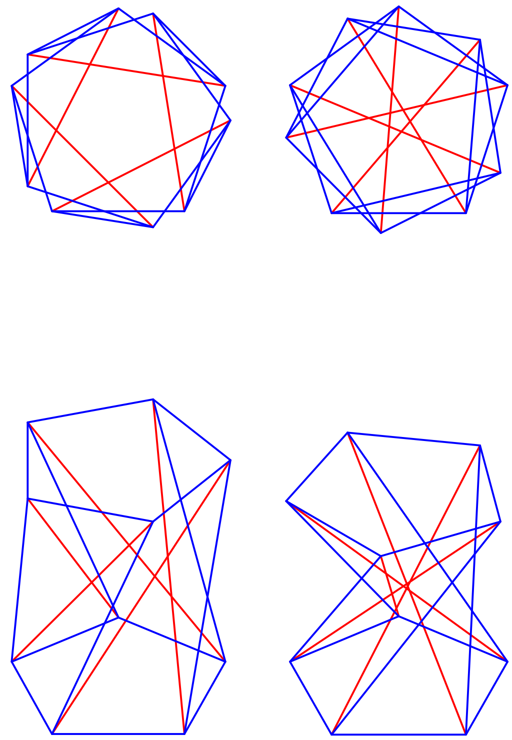

Figure 10

Table 3 shows the values for the axial force on the diagonal elements.

Table 3

f	Numerical
20.0	27.77
-20.0	4.86

Example 4: A circular prismoid with axis on a circumference of radius = 10. The section is defined by a regular triangle inscribed in a circle of radius = 1. It is composed by 72 elements. The modulus of elasticity = 1000. The elements have area = 1. Elements in the initial configuration that start in the state of constant stress are shown in green with tension = 5. There are no support constraints. The loading consists of self-equilibrated radial forces applied on the nodes of the exterior circumference. Due to symmetry, the Tables show information for only one fourth of the structure. Figure 11 shows the initial configuration on the left and the prestressed configuration on the right.

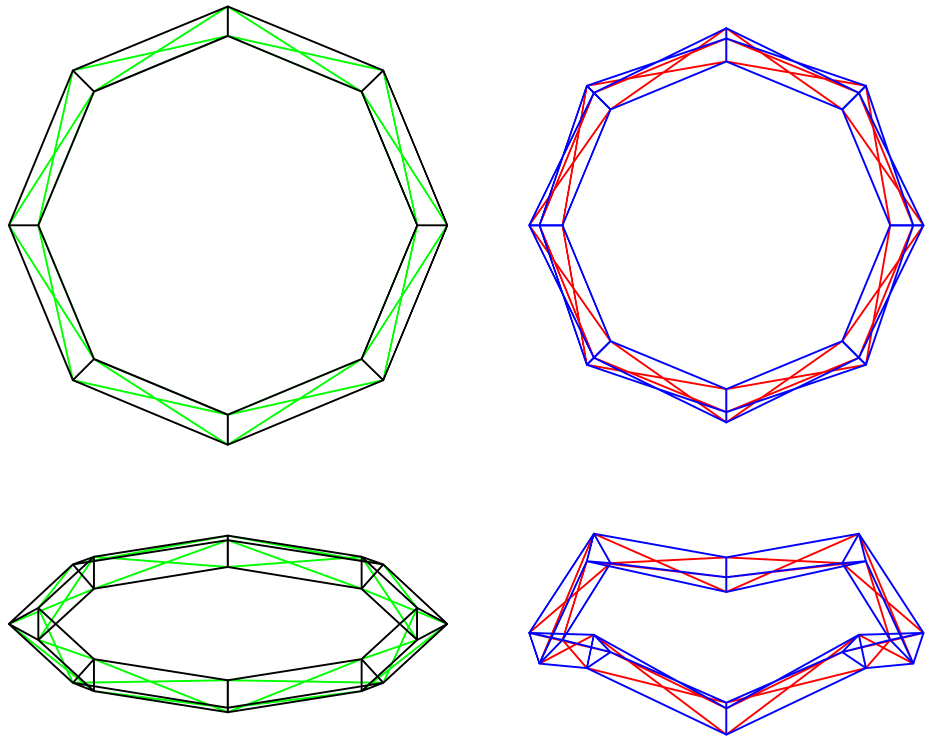

Figure 11

Figure 12 shows the prestressed configuration on the left and the loaded configuration on the right.

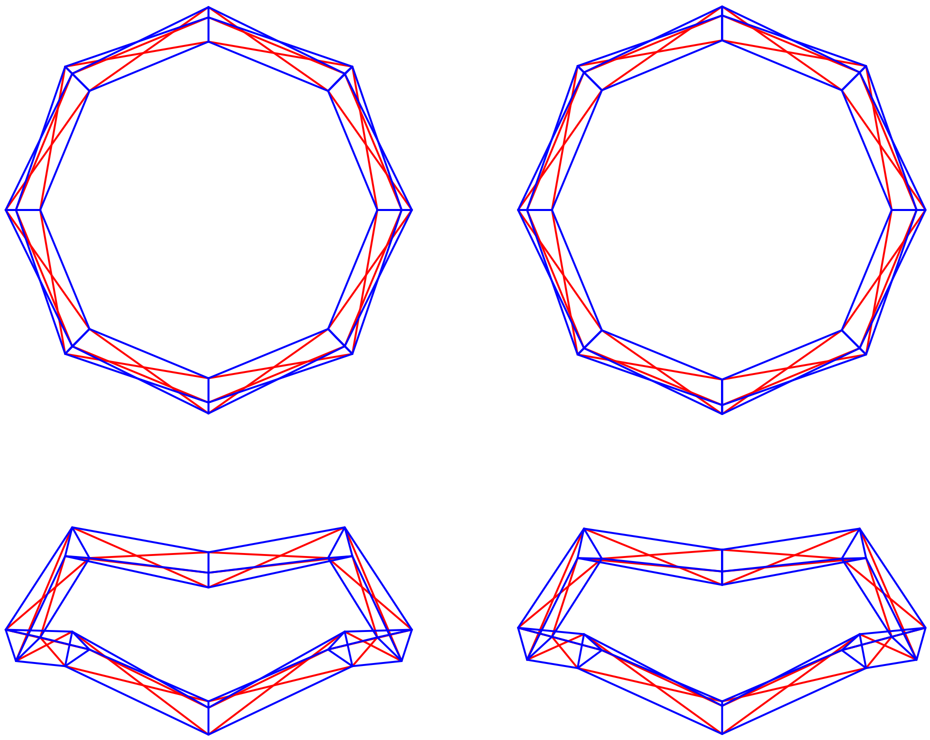

Figure 12

Table 4 shows the coordinates for the initial, the prestressed and loaded configuration respectively.

Table 4

Node	Coord-X	Coord-Y	Coord-Z
1	11.0000	0.0000	0.0000
2	9.5000	0.0000	0.8660
3	9.5000	0.0000	-0.8660
4	7.7782	7.7782	0.0000
5	6.7175	6.7175	0.8660
6	6.7175	6.7175	-0.8660
7	0.0000	11.0000	0.0000
8	0.0000	9.5000	0.8660
9	0.0000	9.5000	-0.8660
Node	Coord-X	Coord-Y	Coord-Z
1	9.3879	-0.0000	-2.1490
2	9.9007	0.0000	-0.4922
3	8.2098	0.0000	-0.8759
4	6.6382	6.6382	2.1490
5	5.8052	5.8052	0.8759
6	7.0009	7.0009	0.4922
7	0.0000	9.3879	-2.1490
8	-0.0000	9.9007	-0.4922
9	0.0000	8.2098	-0.8759

Node	Coord-X	Coord-Y	Coord-Z
1	9.5010	-0.0000	-2.0677
2	9.9371	0.0000	-0.3904
3	8.2630	0.0000	-0.8446
4	6.7182	6.7182	2.0677
5	5.8428	5.8428	0.8446
6	7.0266	7.0266	0.3904
7	0.0000	9.5010	-2.0677
8	-0.0000	9.9371	-0.3904
9	0.0000	8.2630	-0.8446

Table 5 shows the connectivity of the elements.

Table 5

Elem	Node	Node
1	1	2
2	2	3
3	3	1
4	4	5
5	5	6
6	6	4
7	7	8
8	8	9
9	9	7
25	1	4
26	2	5
27	3	6
28	5	8
29	6	9
30	4	7
49	2	4
50	3	5
51	1	6
52	4	8
53	5	9
54	6	7

Table 6 shows the applied forces.

Table 6

Node	Axis	Force
1	1	1.4142
4	1	1.0000
4	2	1.0000
7	2	1.4142

Table 7 shows the axial force for the prestressed and loaded configuration respectively.

Table 7

Elem	Force	Force
1	1.3113	0.5762
2	1.0841	1.5137
3	1.4375	4.7852
4	1.4375	4.7852
5	1.0841	1.5137
6	1.3113	0.5762
7	1.3113	0.5762
8	1.0841	1.5137
9	1.4375	4.7852
25	-5.5232	-6.3702
26	-4.9454	-4.1728
27	-4.9454	-4.1728
28	-4.9454	-4.1728
29	-4.9454	-4.1728
30	-5.5232	-6.3702
49	5.0000	3.7936
50	5.0000	8.5064
51	5.0000	3.7936
52	5.0000	3.7936
53	5.0000	8.5064
54	5.0000	3.7936

Example 5: A Stella Octangula as described in appendix 3 with parameter s = 1. The modulus of elasticity = 1000. The elements have area = 1. There are support constraints on nodes 1, 2 and 3 to prevent rigid body motion. A nonregular tensegrity can be generated by imposing different stress values for selected elements of a regular tensegrity. The regular tensegrity can be recovered by imposing equal stress values for the same selected elements on the previously generated nonregular tensegrity.

The stress values for the diagonal elements of the regular Stella Octangula and the lengths for the diagonal elements of its prestressed configuration are shown in Table 8.

Table 8

Elem	Stress	Length
3	-1.25	1.4573
6	-1.50	1.5664
9	-1.75	1.6312

12	-2.00	1.8578
15	-2.25	1.8899
18	-2.50	1.8914

Figure 13 shows the initial configuration (regular Stella Octangula) on the left and its prestressed configuration (nonregular Stella Octangula) on the right.

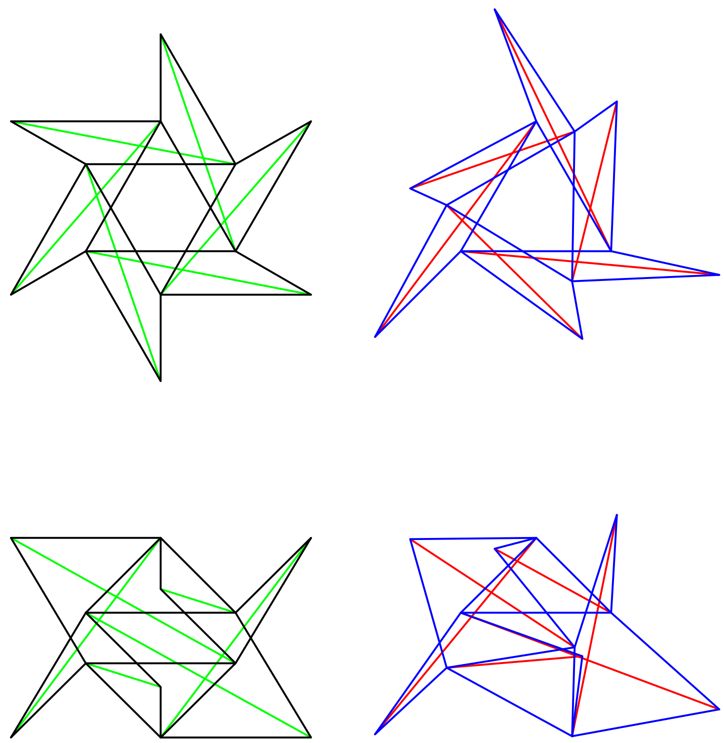

Figure 13

The stress values for the diagonal elements of the nonregular Stella Octangula and the lengths for the diagonal elements of its prestressed configuration are shown in Table 9.

Table 9

Elem	Stress	Length
3	-1.00	1.7343
6	-1.00	1.7345
9	-1.00	1.7348
12	-1.00	1.7351
15	-1.00	1.7353
18	-1.00	1.7357

Figure 14 shows the initial configuration (nonregular Stella Octangula) on the left and its prestressed configuration (regular Stella Octangula) on the right.

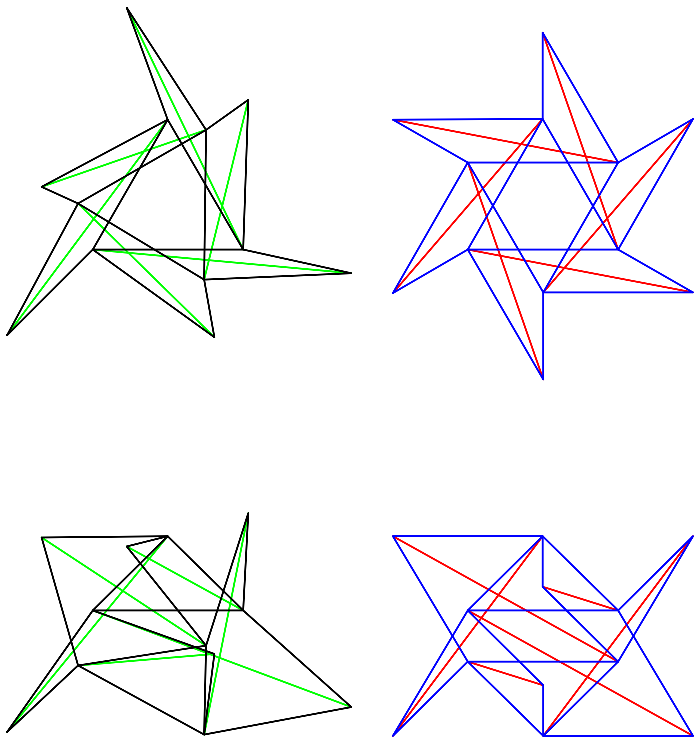

Figure 14

14 Appendix 1

This problem and its relation to tensegrity structures was first described by [1]. Figure 15 shows a two element truss with a vertical displacement on the center node.

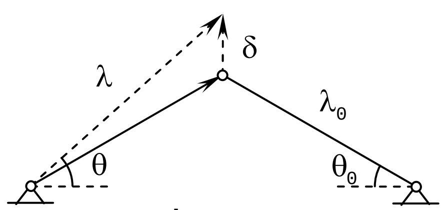

Figure 15

Geometry

$$\lambda \cos\theta = \lambda_0 \cos\theta_0$$

$$\lambda \sin\theta = \lambda_0 \sin\theta_0 + \delta$$

Element length

The element length as function of the rotation angle can be written as:

$$\lambda(\theta) = \frac{\lambda_0 \cos\theta_0}{\cos\theta}$$

The derivative of the rotation angle with respect to the vertical displacement can be written as:

$$\frac{d\theta}{d\delta} = \frac{\cos^2\theta}{\lambda_0 \cos\theta_0}$$

Element strain

Considering a cut μ in the initial length of the element, its undeformed length can be written as:

$$\lambda_\mu = \lambda_0 - \mu$$

The element strain can be written as:

$$\varepsilon = \frac{\lambda(\theta)}{\lambda_\mu} - 1$$

Equilibrium equation

Considering α as the undeformed area of the elements, the total potential strain energy can be written as:

$$\phi = 2\alpha\lambda_\mu \int_0^\varepsilon \sigma(\xi)\, d\xi$$

The derivative of the total potential strain energy with respect to the vertical displacement is equal to the force applied in the direction of this displacement. Note that the force is positive upward.

$$f = \frac{d\phi}{d\varepsilon}\frac{d\varepsilon}{d\theta}\frac{d\theta}{d\delta}$$

$$f = 2\alpha\sigma(\varepsilon)\frac{\cos^2\theta}{\lambda_0\cos\theta_0}\lambda'(\theta)$$

Stress and strain

The following approach can be used when stress is a nonlinear invertible function of strain. For simplicity, consider a linear function with E as the modulus of elasticity. By imposing a tension σ_μ on the elements at the prestressed configuration, its undeformed lengths can be written as:

$$\sigma_\mu = E\left[\frac{\lambda(0)}{\lambda_\mu} - 1\right] \Rightarrow \lambda_\mu = \frac{\lambda_0\cos\theta_0}{\left(1 + \frac{\sigma_\mu}{E}\right)}$$

The cut in the initial length of the elements is given by:

$$\mu = \lambda_0 - \lambda_\mu$$

The equilibrium equation can be written as:

$$\frac{f}{E\alpha} = 2\sin\theta\left[\left(1 + \frac{\sigma_\mu}{E}\right)\frac{1}{\cos\theta} - 1\right]$$

Figure 16 shows the non dimensional force as a function of the rotation angle for $\sigma_\mu/E = 0.001$.

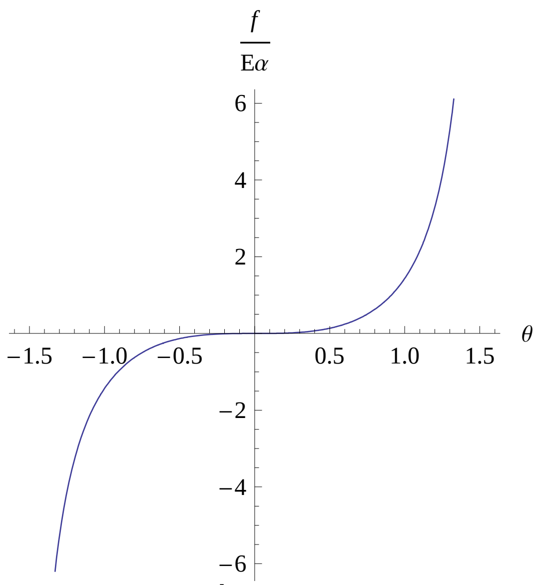

Figure 16

The axial force on the elements can be written as:

$$\alpha\sigma(\varepsilon) = \alpha E\left[\left(1 + \frac{\sigma_\mu}{E}\right)\frac{1}{\cos\theta} - 1\right]$$

15 Appendix 2

The analysis described by [7] for a triangular prismoid tensegrity is extended to a n-sided polygon prismoid tensegrity. Figure 17 shows a straight prismoid. The bottom and top regular polygons are inscribed in circles of equal radius. The sum of the lengths of the diagonal elements (shown in red) is minimized by rotating the top polygon counterclockwise with respect to the bottom polygon, while the lengths of the elements shown in black remain constant.

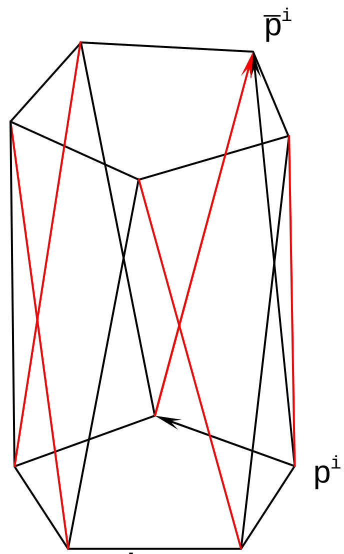

Figure 17

Geometry

For n-sided regular polygons, the coordinates of the vertices can be written as:

$$\gamma = \frac{2\pi}{n}$$

$$p^i = \begin{bmatrix} \rho\cos(\gamma i) \\ \rho\sin(\gamma i) \\ 0 \end{bmatrix}$$

$$\bar{p}^i = \begin{bmatrix} \rho\cos(\theta + \gamma i) \\ \rho\sin(\theta + \gamma i) \\ \delta(\theta) \end{bmatrix}$$

The following vectors are defined in terms of the coordinates of the vertices.

$$b^i = p^{i+1} - p^i$$

$$l^i = \bar{p}^i - p^{i+1}$$

$$v^i = \bar{p}^i - p^i$$

Interval for the rotation angle

Figure 18 shows the top view of a straight prismoid with one diagonal element shown in red.

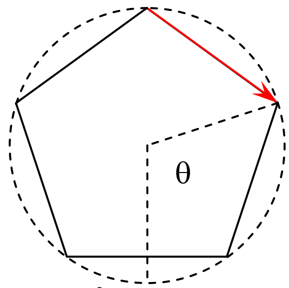

Figure 18

The maximum clockwise rotation happens when a diagonal element intercepts the vertical axis on the center of the circle resulting in diagonal elements interference. This rotation angle is given by:

$$\theta_{min} = -(\pi - \gamma)$$

The maximum counterclockwise rotation happens when a vertical element (connects corresponding vertices of top and bottom polygons) intercepts the vertical axis on the center of the circle resulting in vertical elements interference. This rotation angle is given by:

$$\theta_{max} = \pi$$

The interval for the rotation angle is:

$$-(\pi - \gamma) \leq \theta \leq \pi$$

Height

The square of the norm of vector v^i can be written as:

$$v^i = \overline{p}^i - p^i \Rightarrow$$

$$\left\| v^i \right\|^2 = 2\rho^2 (1 - \cos\theta) + \delta^2 (\theta)$$

Considering ν as the norm of vector v^i, which is constant, the expression for the height as function of rotation angle can be written as:

$$\delta^2 (\theta) = \nu^2 - 2\rho^2 (1 - \cos\theta)$$

Diagonal element length

Considering λ as the norm of vector l^i, the square of the diagonal element length can be written as:

$$b^i + l^i - v^i = 0 \Rightarrow$$

$$\lambda^2(\theta) = \nu^2 + 2\rho^2\left[\cos\theta - \cos(\theta - \gamma)\right]$$

Minimum diagonal element length

Due to symmetry, minimizing the sum of the diagonal element lengths is equivalent to minimizing the square of one diagonal element length.

$$\frac{\partial\lambda^2}{\partial\theta} = 0 \Rightarrow \tan\bar{\theta} = \frac{\sin\gamma}{\cos\gamma - 1}$$

Notice that this expression is valid when the diagonal elements connect the corresponding bottom and top points in any symmetric way.

$$\bar{\theta} > 0 \Rightarrow$$

$$\cos\bar{\theta} = \frac{\cos\gamma - 1}{\sqrt{2(1 - \cos\gamma)}}$$

$$\sin\bar{\theta} = \frac{\sin\gamma}{\sqrt{2(1 - \cos\gamma)}}$$

Element strain

Considering a cut μ in the initial length of the diagonal element, its undeformed length can be written as:

$$\lambda_\mu = \lambda(0) - \mu$$

The element strain can be written as:

$$\varepsilon = \frac{\lambda(\theta)}{\lambda_\mu} - 1$$

Equilibrium equation

Considering α as the undeformed area of the diagonal elements, the total potential strain energy can be written as:

$$\phi = n\alpha\lambda_\mu \int_0^\varepsilon \sigma(\xi)\, d\xi$$

The derivative of the total potential strain energy with respect to the vertical displacement is equal to the force applied in the direction of this displacement. This derivative is equal to the derivative with respect to the height. Note that the force is positive upward when the vertical displacements of the bottom vertices are fixed.

$$f = \frac{d\phi}{d\varepsilon}\frac{d\varepsilon}{d\theta}\frac{d\theta}{d\delta}$$

$$f = n\alpha\sigma(\varepsilon)\left[1 - \frac{\sin(\theta - \gamma)}{\sin\theta}\right]\frac{\delta(\theta)}{\lambda(\theta)}$$

Stress and strain

The following approach can be used when stress is a nonlinear invertible function of strain. For simplicity, consider a linear function with E as the modulus of elasticity. By imposing a tension σ_μ on the diagonal elements at the prestressed configuration, its undeformed lengths can be written as:

$$\sigma_\mu = E\left[\frac{\lambda(\overline{\theta})}{\lambda_\mu} - 1\right] \Rightarrow \lambda_\mu = \frac{\sqrt{\nu^2 - 2\rho^2\sqrt{2(1 - \cos\gamma)}}}{\left(1 + \frac{\sigma_\mu}{E}\right)}$$

The cut in the initial length of the diagonal elements can be written as:

$$\mu = \lambda(0) - \lambda_\mu$$

The equilibrium equation can be written as:

$$\frac{f}{E\alpha} = n\left[\frac{1}{\lambda_\mu} - \frac{1}{\lambda(\theta)}\right]\left[1 - \frac{\sin(\theta-\gamma)}{\sin\theta}\right]\delta(\theta)$$

Figure 19 shows the non dimensional force as a function of the rotation angle for n = 5, $\nu/\rho = 3$ and $\sigma_\mu/E = 0.001$. Note that the vertical axis is placed on the position defined by the prestressed configuration.

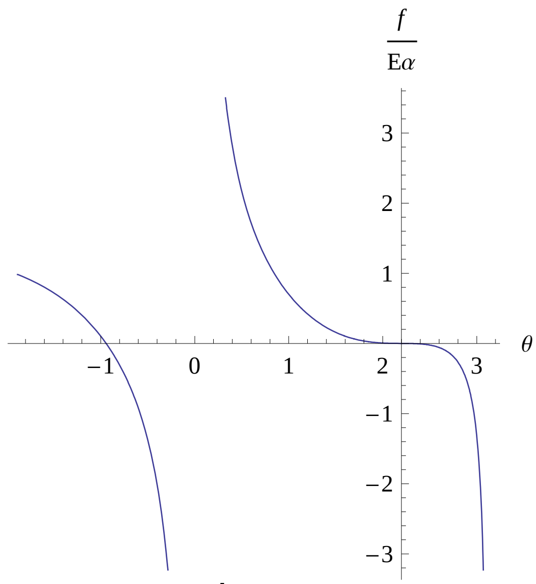

Figure 19

The axial force on the diagonal elements can be written as:

$$\alpha\sigma(\varepsilon) = \alpha E\left[\frac{\lambda(\theta)}{\lambda_\mu} - 1\right]$$

16 Appendix 3

Figure 20 shows the geometry of a sculpture called Stella Octangula, which was proposed by David Georges Emmerich. He was a Hungarian architect, sculptor and author. An extensive description of his works is given by [2]. An analysis of this structure, using the dynamic relaxation method, was described by [5].

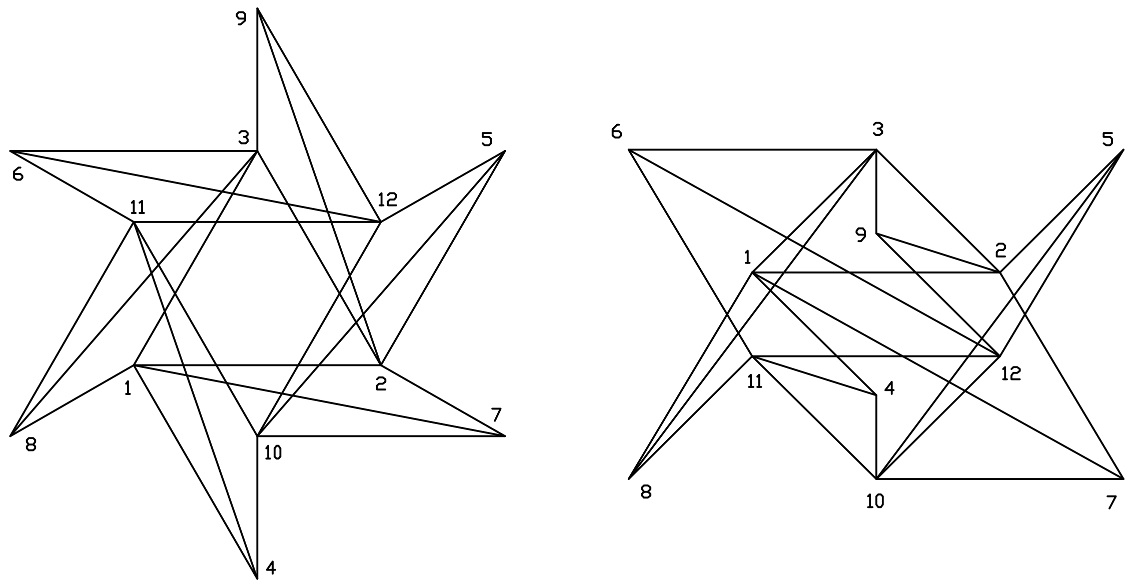

Figure 20

The geometry is composed by 18 elements with length equal to s and 6 diagonal elements with length equal to $s\sqrt{3}$. Table 10 shows the coordinates of the vertices, where the parameters r and h are given by:

$$r = \frac{s}{\sqrt{3}}$$

$$h = \frac{s}{\sqrt{6}}$$

Table 10

Node	Coord-X	Coord-Y	Coord-Z
1	-s/2	-r/2	h
2	s/2	-r/2	h
3	0	r	h
4	0	-2r	h
5	s	r	h
6	-s	r	h
7	s	-r	-h
8	-s	-r	-h
9	0	2r	-h
10	0	-r	-h
11	-s/2	r/2	-h
12	s/2	r/2	-h

Table 11 shows the connectivity of the diagonal elements.

Table 11

Elem	Node	Node
3	4	11
6	5	10
9	6	12
12	7	1
15	8	3
18	9	2

17 References

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