FINITE ELEMENT ANALYSIS OF 3D ORTHOTROPIC MEMBRANE STRUCTURES

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This text derives a mathematical model for a 3D orthotropic membrane finite element. It consists of a total Lagrangian description of a linear elastic material, and can be used to calculate either the Green strain or the engineering strain. The total potential energy is minimized using a quasi-Newton method, making it unnecessary to calculate the stiffness matrix. The source and executable computer codes of the algorithm are available from the author's website.

Keywords: element, membrane, minimization, nonlinear, optimization, orthotropic, triangular.

1 Notation

The following conventions apply unless otherwise specified or made clear by the context. A Greek letter expresses a scalar. A lower case letter represents a column vector. An upper case letter represents a matrix.

2 Introduction

The approach used in this text recovers the basic idea of minimizing the total potential energy to find equilibrium. In the context of tension structures, this idea was first introduced by Coyette and Guisset [1988] for cable network analysis. As the total potential energy is a nonlinear function of the nodal displacements, a quasi-Newton method is used to find its minimum. The advantages of this approach are: It is not necessary to derive an expression for the stiffness matrix, it is not necessary to solve any system of equations, and it permits a simple static analysis instead of a pseudo-dynamic analysis, such as dynamic relaxation with kinetic damping as described by Barnes [1999]. The computer code uses the limited memory BFGS to tackle large scale problems as described by Nocedal and Wright [1999]. It also employs a line search procedure with safeguards as described by Gill and Murray [1974].

3 Finite element definition

Figure 1 shows a reference system with the xy plane located in the plane of the element. The nodes are labeled 1, 2 and 3 while traversing the sides in counter-clockwise fashion. The respective internal angles are labeled α_1, α_2 and α_3. Each side is labeled with the number of its opposite node. The x axis is chosen parallel to side 3 without loss of generality. The strains are assumed to be constant over the element.

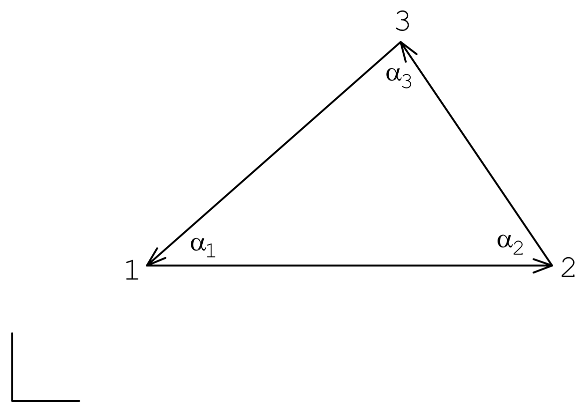

Figure 1

4 Stress strain relations

Considering Figure 2, the transformation of strain and stress can be written respectively as follows:

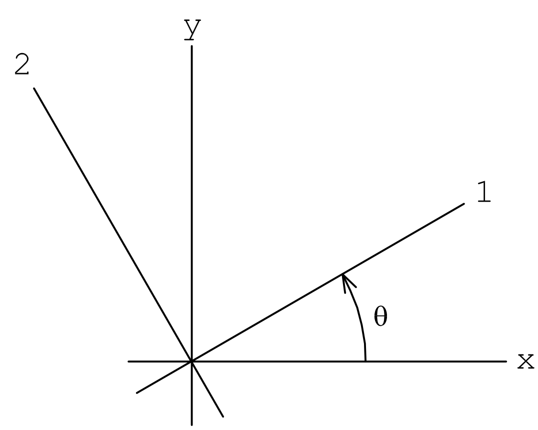

Figure 2

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & \left(c^2 - s^2\right) \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} \Rightarrow \hat{\varepsilon} = T\bar{\varepsilon}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & \left(c^2 - s^2\right) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} \Rightarrow \hat{\sigma} = T\bar{\sigma}$$

where,

$$c = \cos\theta \quad , \quad s = \sin\theta$$

$$\hat{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \sqrt{2}\varepsilon_{12} \end{bmatrix} \quad , \quad \bar{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \sqrt{2}\varepsilon_{xy} \end{bmatrix}$$

$$\hat{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sqrt{2}\sigma_{12} \end{bmatrix} \quad , \quad \bar{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sqrt{2}\sigma_{xy} \end{bmatrix}$$

$$T = \begin{bmatrix} c^2 & s^2 & \sqrt{2}cs \\ s^2 & c^2 & -\sqrt{2}cs \\ -\sqrt{2}cs & \sqrt{2}cs & \left(c^2 - s^2\right) \end{bmatrix}$$

The stress strain relations for the material directions can be written as follows:

$$\hat{\sigma} = \hat{H}\hat{\varepsilon}$$

where,

$$\hat{H} = \begin{bmatrix} \dfrac{E_1}{\left(1 - \nu_{12}\nu_{21}\right)} & \dfrac{\nu_{12}E_2}{\left(1 - \nu_{12}\nu_{21}\right)} & 0 \\ \dfrac{\nu_{12}E_2}{\left(1 - \nu_{12}\nu_{21}\right)} & \dfrac{E_2}{\left(1 - \nu_{12}\nu_{21}\right)} & 0 \\ 0 & 0 & 2G_{12} \end{bmatrix}$$

ν_{12} : Poisson's ratio for strain in direction 2 when stressed in direction 1 only.

G_{12} : Shear modulus in the 1,2 plane.

$$\nu_{21}E_1 = \nu_{12}E_2$$

The stress strain relations for arbitrary directions, where the material directions have been rotated a positive angle θ from the x-axis, can be written as follows:

$$\hat{\sigma} = \hat{H}\hat{\varepsilon}$$

$$\hat{\varepsilon} = T\overline{\varepsilon} \Rightarrow \hat{\sigma} = \hat{H}T\overline{\varepsilon}$$

$$\hat{\sigma} = T\overline{\sigma} \Rightarrow T\overline{\sigma} = \hat{H}T\overline{\varepsilon} \Rightarrow \overline{\sigma} = T^{-1}\hat{H}T\overline{\varepsilon}$$

$$TT^T = I \Rightarrow \overline{\sigma} = \overline{H}\overline{\varepsilon}$$

Where,

$$\overline{H} = T^T\hat{H}T$$

5 Directional strain

The strain of an infinitesimal line segment in the direction of a unitary vector u^i, can be written as:

$$\varepsilon_i = c_i^2\varepsilon_{xx} + s_i^2\varepsilon_{yy} + 2c_is_i\varepsilon_{xy}$$

where,

$$u^i = \begin{bmatrix} c_i \\ s_i \end{bmatrix} \quad , \quad c_i = \cos\theta_i \quad , \quad s_i = \sin\theta_i$$

Considering Figure 1, the directional strains for the sides of the triangle can be written as follows. First, note the trigonometric relations

$$\theta_1 + \alpha_2 = \pi \Rightarrow \begin{cases} c_1 = -\cos\alpha_2 \\ s_1 = +\sin\alpha_2 \end{cases}$$

and

$$\theta_2 = \pi + \alpha_1 \Rightarrow \begin{cases} c_2 = -\cos\alpha_1 \\ s_2 = -\sin\alpha_1 \end{cases}$$

The three strains are therefore

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} \cos^2\alpha_2 & \sin^2\alpha_2 & -\sqrt{2}\cos\alpha_2\sin\alpha_2 \\ \cos^2\alpha_1 & \sin^2\alpha_1 & +\sqrt{2}\cos\alpha_1\sin\alpha_1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \sqrt{2}\varepsilon_{xy} \end{bmatrix} \Rightarrow \varepsilon = C\bar{\varepsilon}$$

where,

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \;,\; C = \begin{bmatrix} \cos^2\alpha_2 & \sin^2\alpha_2 & -\sqrt{2}\cos\alpha_2\sin\alpha_2 \\ \cos^2\alpha_1 & \sin^2\alpha_1 & +\sqrt{2}\cos\alpha_1\sin\alpha_1 \\ 1 & 0 & 0 \end{bmatrix}$$

It is easy to show that,

$$|C| = \sqrt{2}\sin\alpha_1\sin\alpha_2\sin\alpha_3$$

and that

$$C^{-1} = \frac{1}{|C|}\begin{bmatrix} 0 & 0 & \sqrt{2}\sin\alpha_1\sin\alpha_2\sin\alpha_3 \\ \sqrt{2}\cos\alpha_1\sin\alpha_1 & \sqrt{2}\cos\alpha_2\sin\alpha_2 & -\sqrt{2}\cos\alpha_1\cos\alpha_2\sin\alpha_3 \\ -\sin^2\alpha_1 & \sin^2\alpha_2 & \sin(\alpha_1 - \alpha_2)\sin\alpha_3 \end{bmatrix}$$

6 Strain energy density

The strain energy density for a linearly elastic body can be written as

$$\varphi = \frac{1}{2}\bar{\varepsilon}^T\bar{\sigma}$$

where,

$$\bar{\sigma} = \bar{H}\bar{\varepsilon} \Rightarrow \varphi = \frac{1}{2}\bar{\varepsilon}^T\bar{H}\bar{\varepsilon}$$

This can be written in terms of the directional strains by the following steps:

$$\varepsilon = C\bar{\varepsilon} \Rightarrow \bar{\varepsilon} = C^{-1}\varepsilon \Rightarrow \varphi = \frac{1}{2}\varepsilon^{T}H\varepsilon = \varphi\left(\varepsilon_1, \varepsilon_2, \varepsilon_3\right)$$

where,

$$H = C^{-T}\bar{H}C^{-1}$$

$$\bar{H} = T^{T}\hat{H}T \Rightarrow H = \left(TC^{-1}\right)^{T}\hat{H}\left(TC^{-1}\right)$$

6.1 Potential strain energy

Considering v as the undeformed volume of the element, the potential strain energy can be written as

$$\phi = \int_{v} \varphi\left(\varepsilon_1, \varepsilon_2, \varepsilon_3\right) dv$$

Considering α as the undeformed area of the element and t its undeformed thickness, the potential strain energy can be written as

$$\phi = \frac{1}{2}\varepsilon^{T}\left(Ht\right)\varepsilon\alpha = \frac{1}{2}\varepsilon^{T}\left(TC^{-1}\right)^{T}\left(\hat{H}t\right)\left(TC^{-1}\right)\varepsilon\alpha$$

Note that the product of matrix H by the element's thickness can be achieved by multiplying the Young's modulus and the Shear modulus by the element's thickness. The Young's modulus and the Shear modulus can be thought as having the dimension of force by length.

$$\hat{H}t = \begin{bmatrix} \frac{\left(E_1 t\right)}{\left(1 - \nu_{12}\nu_{21}\right)} & \frac{\nu_{12}\left(E_2 t\right)}{\left(1 - \nu_{12}\nu_{21}\right)} & 0 \\ \frac{\nu_{12}\left(E_2 t\right)}{\left(1 - \nu_{12}\nu_{21}\right)} & \frac{\left(E_2 t\right)}{\left(1 - \nu_{12}\nu_{21}\right)} & 0 \\ 0 & 0 & 2\left(G_{12}t\right) \end{bmatrix}$$

6.2 Gradient of the potential strain energy

The gradient of the potential strain energy can be written as follows:

$$\phi = \int_v \varphi\left(\varepsilon_1, \varepsilon_2, \varepsilon_3\right) dv \Rightarrow \frac{\partial\phi}{\partial x_i} = \left(t\,\frac{\partial\varphi}{\partial\varepsilon_1}\,\frac{\partial\varepsilon_1}{\partial x_i} + t\,\frac{\partial\varphi}{\partial\varepsilon_2}\,\frac{\partial\varepsilon_2}{\partial x_i} + t\,\frac{\partial\varphi}{\partial\varepsilon_3}\,\frac{\partial\varepsilon_3}{\partial x_i}\right)\alpha$$

$$\varphi = \frac{1}{2}\,\varepsilon^T H\varepsilon \Rightarrow \begin{bmatrix} t\,\dfrac{\partial\varphi}{\partial\varepsilon_1} \\ t\,\dfrac{\partial\varphi}{\partial\varepsilon_2} \\ t\,\dfrac{\partial\varphi}{\partial\varepsilon_3} \end{bmatrix} = (Ht)\,\varepsilon$$

$$\sigma = (Ht)\,\varepsilon \Rightarrow \frac{\partial\phi}{\partial x_i} = \left(\sigma^T\,\frac{\partial\varepsilon}{\partial x_i}\right)\alpha$$

Note that the stress has been multiplied by the element's thickness. The stress can be thought as having the dimension of force by length.

7 Strain components and its derivatives

The nodal displacements vectors are numbered according to its node numbers as shown in Figure 3.

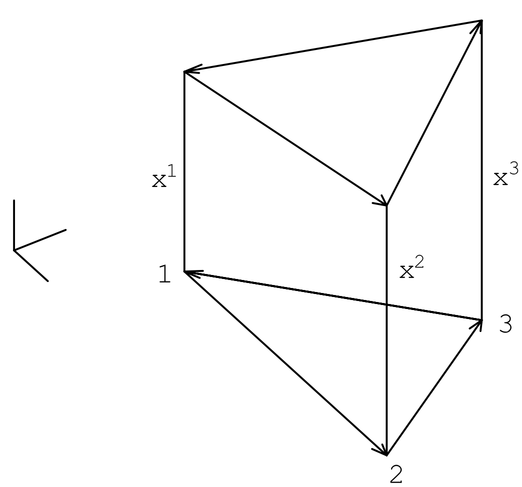

Figure 3

Their individual components are numbered as follows:

$$x^1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;,\; x^2 = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} \;,\; x^3 = \begin{bmatrix} x_7 \\ x_8 \\ x_9 \end{bmatrix}$$

To write the directional strain for a side of the triangle consider Figure 4, where u is a unitary vector parallel to the undeformed side, λ is the undeformed length of the side and p and q are the nodal displacements vectors.

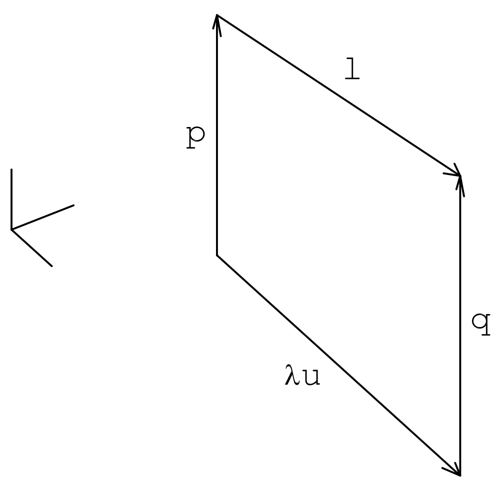

Figure 4

$$\lambda u + q - l - p = 0$$

$$l = \lambda u + q - p$$

Defining

$$z = \frac{q - p}{\lambda}$$

it follows that

$$l = \lambda (u + z)$$

Defining

$$\delta = 2u^T z + z^T z$$

it follows that

$$l^T l = \lambda^2 (1 + \delta)$$

More generally, consider u^k as a unitary vector parallel to the undeformed side k and λ_k as undeformed length of side k.

7.1 Engineering strain

The Engineering strain along the side of the element and its derivatives with respect to the nodal displacements can be written as follows:

$$\varepsilon = \frac{\sqrt{l^T l} - \lambda}{\lambda} = \sqrt{1 + \delta} - 1$$

$$\frac{\partial \varepsilon}{\partial p_i} = -\frac{1}{\lambda\sqrt{1 + \delta}} (u_i + z_i)$$

$$\frac{\partial \varepsilon}{\partial q_i} = +\frac{1}{\lambda\sqrt{1 + \delta}} (u_i + z_i)$$

The expressions for side 1 can be written as follows:

$$z = \frac{x^3 - x^2}{\lambda_1} \quad , \quad \delta = 2\left(u^1\right)^T z + z^T z \quad , \quad \varepsilon_1 = \frac{\delta}{\sqrt{1 + \delta} + 1}$$

$$\nabla \varepsilon_1 = \frac{1}{\lambda_1 \sqrt{1 + \delta}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\left(u_1^1 + z_1^1\right) \\ -\left(u_2^1 + z_2^1\right) \\ -\left(u_3^1 + z_3^1\right) \\ +\left(u_1^1 + z_1^1\right) \\ +\left(u_2^1 + z_2^1\right) \\ +\left(u_3^1 + z_3^1\right) \end{bmatrix}$$

The expressions for side 2 can be written as follows:

$$z = \frac{x^1 - x^3}{\lambda_2} \quad , \quad \delta = 2\left(u^2\right)^T z + z^T z \quad , \quad \varepsilon_2 = \frac{\delta}{\sqrt{1 + \delta} + 1}$$

$$\nabla\varepsilon_2 = \frac{1}{\lambda_2\sqrt{1 + \delta}} \begin{bmatrix} +\left(u_1^2 + z_1^2\right) \\ +\left(u_2^2 + z_2^2\right) \\ +\left(u_3^2 + z_3^2\right) \\ 0 \\ 0 \\ 0 \\ -\left(u_1^2 + z_1^2\right) \\ -\left(u_2^2 + z_2^2\right) \\ -\left(u_3^2 + z_3^2\right) \end{bmatrix}$$

The expressions for side 3 can be written as follows:

$$z = \frac{x^2 - x^1}{\lambda_3} \quad , \quad \delta = 2\left(u^3\right)^T z + z^T z \quad , \quad \varepsilon_3 = \frac{\delta}{\sqrt{1 + \delta} + 1}$$

$$\nabla\varepsilon_3 = \frac{1}{\lambda_3\sqrt{1 + \delta}} \begin{bmatrix} -\left(u_1^3 + z_1^3\right) \\ -\left(u_2^3 + z_2^3\right) \\ -\left(u_3^3 + z_3^3\right) \\ +\left(u_1^3 + z_1^3\right) \\ +\left(u_2^3 + z_2^3\right) \\ +\left(u_3^3 + z_3^3\right) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

7.2 Green strain

The Green strain along the side of the element and its derivatives with respect to the nodal displacements can be written as follows:

$$\varepsilon = \frac{l^T l - \lambda^2}{2\lambda^2} = \frac{\delta}{2}$$

$$\frac{\partial \varepsilon}{\partial p_i} = -\frac{1}{\lambda}\left(u_i + z_i\right)$$

$$\frac{\partial \varepsilon}{\partial q_i} = +\frac{1}{\lambda}\left(u_i + z_i\right)$$

The expressions for side 1 can be written as follows:

$$z = \frac{x^3 - x^2}{\lambda_1} \quad , \quad \delta = 2\left(u^1\right)^T z + z^T z\,, \quad \varepsilon_1 = \frac{\delta}{2}$$

$$\nabla \varepsilon_1 = \frac{1}{\lambda_1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ -\left(u_1^1 + z_1^1\right) \\ -\left(u_2^1 + z_2^1\right) \\ -\left(u_3^1 + z_3^1\right) \\ +\left(u_1^1 + z_1^1\right) \\ +\left(u_2^1 + z_2^1\right) \\ +\left(u_3^1 + z_3^1\right) \end{bmatrix}$$

The expressions for side 2 can be written as follows:

$$z = \frac{x^1 - x^3}{\lambda_2} \quad , \quad \delta = 2\left(u^2\right)^T z + z^T z\,, \quad \varepsilon_2 = \frac{\delta}{2}$$

$$\nabla \varepsilon_2 = \frac{1}{\lambda_2}\begin{bmatrix} +\left(u_1^2 + z_1^2\right) \\ +\left(u_2^2 + z_2^2\right) \\ +\left(u_3^2 + z_3^2\right) \\ 0 \\ 0 \\ 0 \\ -\left(u_1^2 + z_1^2\right) \\ -\left(u_2^2 + z_2^2\right) \\ -\left(u_3^2 + z_3^2\right) \end{bmatrix}$$

The expressions for side 3 can be written as follows:

$$z = \frac{x^2 - x^1}{\lambda_3}\ , \quad \delta = 2\left(u^3\right)^T z + z^T z\, , \quad \varepsilon_3 = \frac{\delta}{2}$$

$$\nabla\varepsilon_3 = \frac{1}{\lambda_3}\begin{bmatrix} -\left(u_1^3 + z_1^3\right) \\ -\left(u_2^3 + z_2^3\right) \\ -\left(u_3^3 + z_3^3\right) \\ +\left(u_1^3 + z_1^3\right) \\ +\left(u_2^3 + z_2^3\right) \\ +\left(u_3^3 + z_3^3\right) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

8 Equilibrium configurations

The stable equilibrium configurations correspond to local minimum points of the total potential energy function. It is advisable to use a Quasi Newton type method to find these local minimums because it does not require the evaluation of the stiffness matrix.

Considering x as the vector of unknown displacements and f as the vector of nodal forces, the total potential energy function π and its gradient can be written as follows:

$$\pi(x) = \sum_{\text{elements}} \phi(x) - f^T x$$

$$\nabla\pi(x) = \sum_{\text{elements}} \nabla\phi(x) - f$$

8.1 Pressure as follower forces

A sequence of major iterations is employed to treat pressure as follower forces. At each of these major iterations, the pressure is applied as fixed forces on nodes, orthogonal to the element's surface. In general, at the start of each major iteration a loading update is performed by applying the pressure on the deformed configuration obtained in the previous major iteration. The exception is the first major iteration, where the pressure is applied on the undeformed structure. At each major iteration, the equilibrium is obtained through a sequence of minor iteration that minimizes the total potential energy.

9 Principal stresses

The stresses for the reference system shown in Figure 1 can be written as

$$\sigma = H\varepsilon$$

$$H = C^{-T}\overline{H}C^{-1} \Rightarrow \sigma = C^{-T}\overline{H}C^{-1}\varepsilon$$

$$\varepsilon = C\overline{\varepsilon} \Rightarrow \sigma = C^{-T}\overline{H}\overline{\varepsilon}$$

$$\overline{\sigma} = \overline{H}\overline{\varepsilon} \Rightarrow \sigma = C^{-T}\overline{\sigma} \Rightarrow \overline{\sigma} = C^{T}\sigma$$

The principal stresses can be written as:

$$\overline{\sigma} = C^{T}\sigma \Rightarrow \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \cos^2\alpha_2 & \cos^2\alpha_1 & 1 \\ \sin^2\alpha_2 & \sin^2\alpha_1 & 0 \\ -\cos\alpha_2\sin\alpha_2 & +\cos\alpha_1\sin\alpha_1 & 0 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}$$

$$\Delta = \left(\sigma_{xx} - \sigma_{yy}\right)^2 + 4\sigma_{xy}^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 +$$
$$+4\sigma_1\sigma_2\left(\cos^2\alpha_3 - \frac{1}{2}\right) + 4\sigma_1\sigma_3\left(\cos^2\alpha_2 - \frac{1}{2}\right) + 4\sigma_2\sigma_3\left(\cos^2\alpha_1 - \frac{1}{2}\right)$$

$$\sigma' = \frac{\left(\sigma_{xx} + \sigma_{yy}\right) \pm \sqrt{\Delta}}{2} = \frac{\left(\sigma_1 + \sigma_2 + \sigma_3\right) \pm \sqrt{\Delta}}{2}$$

10 Appendix

10.1 Transformation of strain – 2D

Consider Figure 5 that shows a reference system xy which has been rotated by an angle θ from the reference system xy. Note that $\varepsilon_{\mathit{xx}}$ can be interpreted as the strain in the direction of a unit vector parallel to the x axis. The transformation is

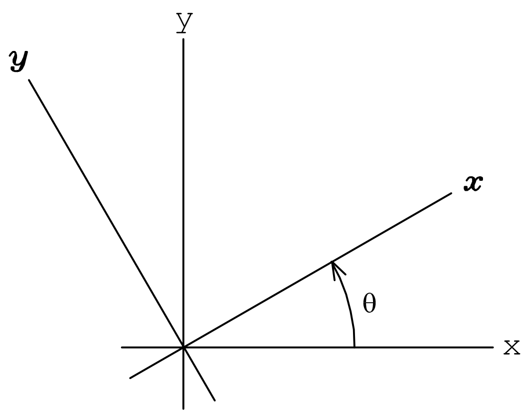

Figure 5

$$\left.\begin{aligned}\mathrm{x} &= \mathit{x}\cos\theta - \mathit{y}\sin\theta \\ \mathrm{y} &= \mathit{x}\sin\theta + \mathit{y}\cos\theta\end{aligned}\right\} \Rightarrow \begin{bmatrix}\mathrm{x}\\ \mathrm{y}\end{bmatrix} = \begin{bmatrix}+c & -s\\ +s & +c\end{bmatrix}\begin{bmatrix}\mathit{x}\\ \mathit{y}\end{bmatrix}$$

where,

$$c = \cos\theta \quad , \quad s = \sin\theta$$

When a body is deformed, the point (x,y) is displaced to the point (x + u,y + v), where (u,v) denotes the components of the displacement.

$$\begin{bmatrix}\mathrm{u}\\ \mathrm{v}\end{bmatrix} = \begin{bmatrix}+c & -s\\ +s & +c\end{bmatrix}\begin{bmatrix}\mathit{x}+\mathit{u}\\ \mathit{y}+\mathit{v}\end{bmatrix} - \begin{bmatrix}+c & -s\\ +s & +c\end{bmatrix}\begin{bmatrix}\mathit{x}\\ \mathit{y}\end{bmatrix}$$

$$\begin{bmatrix}\mathrm{u}\\ \mathrm{v}\end{bmatrix} = \begin{bmatrix}+c & -s\\ +s & +c\end{bmatrix}\begin{bmatrix}\mathit{u}\\ \mathit{v}\end{bmatrix} \Rightarrow \begin{bmatrix}\mathit{u}\\ \mathit{v}\end{bmatrix} = \begin{bmatrix}+c & +s\\ -s & +c\end{bmatrix}\begin{bmatrix}\mathrm{u}\\ \mathrm{v}\end{bmatrix}$$

$$\mathit{u} = c\mathrm{u} + s\mathrm{v}$$

$$\frac{\partial \mathit{u}}{\partial \mathit{x}} = c\left(\frac{\partial \mathrm{u}}{\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial \mathit{x}} + \frac{\partial \mathrm{u}}{\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial \mathit{x}}\right) + s\left(\frac{\partial \mathrm{v}}{\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial \mathit{x}} + \frac{\partial \mathrm{v}}{\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial \mathit{x}}\right)$$

$$\frac{\partial u}{\partial x} = \mathrm{c}^2 \frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{s}^2 \frac{\partial \mathrm{v}}{\partial \mathrm{y}} + \mathrm{cs}\left(\frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \frac{\partial \mathrm{v}}{\partial \mathrm{x}}\right)$$

$$\frac{\partial u}{\partial y} = \mathrm{c}\left(\frac{\partial \mathrm{u}}{\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial y} + \frac{\partial \mathrm{u}}{\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial y}\right) + \mathrm{s}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial y} + \frac{\partial \mathrm{v}}{\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial y}\right)$$

$$\frac{\partial u}{\partial y} = -\mathrm{cs}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{c}^2 \frac{\partial \mathrm{u}}{\partial \mathrm{y}} - \mathrm{s}^2 \frac{\partial \mathrm{v}}{\partial \mathrm{x}} + \mathrm{cs}\frac{\partial \mathrm{v}}{\partial \mathrm{y}}$$

$$v = -\mathrm{su} + \mathrm{cv}$$

$$\frac{\partial v}{\partial x} = -\mathrm{s}\left(\frac{\partial \mathrm{u}}{\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial x} + \frac{\partial \mathrm{u}}{\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial x}\right) + \mathrm{c}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial x} + \frac{\partial \mathrm{v}}{\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial x}\right)$$

$$\frac{\partial v}{\partial x} = -\mathrm{cs}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} - \mathrm{s}^2 \frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \mathrm{c}^2 \frac{\partial \mathrm{v}}{\partial \mathrm{x}} + \mathrm{cs}\frac{\partial \mathrm{v}}{\partial \mathrm{y}}$$

$$\frac{\partial v}{\partial y} = -\mathrm{s}\left(\frac{\partial \mathrm{u}}{\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial y} + \frac{\partial \mathrm{u}}{\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial y}\right) + \mathrm{c}\left(\frac{\partial \mathrm{v}}{\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial y} + \frac{\partial \mathrm{v}}{\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial y}\right)$$

$$\frac{\partial v}{\partial y} = \mathrm{s}^2 \frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{c}^2 \frac{\partial \mathrm{v}}{\partial \mathrm{y}} - \mathrm{cs}\left(\frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \frac{\partial \mathrm{v}}{\partial \mathrm{x}}\right)$$

10.1.1 Engineering strain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

The strain xx can be written as follows:

$$\frac{\partial u}{\partial x} = \mathrm{c}^2 \frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{s}^2 \frac{\partial \mathrm{v}}{\partial \mathrm{y}} + \mathrm{cs}\left(\frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \frac{\partial \mathrm{v}}{\partial \mathrm{x}}\right)$$

$$\varepsilon_{xx} = c^2\varepsilon_{xx} + s^2\varepsilon_{yy} + 2cs\varepsilon_{xy}$$

The strain yy can be written as follows:

$$\frac{\partial v}{\partial y} = s^2 \frac{\partial u}{\partial x} + c^2 \frac{\partial v}{\partial y} - cs\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\varepsilon_{yy} = s^2\varepsilon_{xx} + c^2\varepsilon_{yy} - 2cs\varepsilon_{xy}$$

The strain xy can be written as follows:

$$\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -cs\frac{\partial u}{\partial x} + cs\frac{\partial v}{\partial y} + \left(c^2 - s^2\right)\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\varepsilon_{xy} = -cs\varepsilon_{xx} + cs\varepsilon_{yy} + \left(c^2 - s^2\right)\varepsilon_{xy}$$

The transformation for the three strain components can be written in matrix form as

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & \left(c^2 - s^2\right) \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix}$$

10.1.2 Green strain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right)$$

The strain xx can be written as follows:

$$\frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] =$$

$$+c^2\left\{\frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]\right\} +$$

$$+s^2\left\{\frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]\right\} +$$

$$+cs\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right)$$

$$\varepsilon_{xx} = c^2\varepsilon_{xx} + s^2\varepsilon_{yy} + 2cs\varepsilon_{xy}$$

The strain yy can be written as follows:

$$\frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] =$$

$$+s^2\left\{\frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]\right\} +$$

$$+c^2\left\{\frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]\right\} +$$

$$-cs\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right)$$

$$\varepsilon_{yy} = s^2\varepsilon_{xx} + c^2\varepsilon_{yy} - 2cs\varepsilon_{xy}$$

The strain xy can be written as follows:

$$\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right) =$$

$$-cs\left\{\frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]\right\} +$$

$$+cs\left\{\frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]\right\} +$$

$$+\left(c^2 - s^2\right)\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right)$$

$$\varepsilon_{xy} = -cs\varepsilon_{xx} + cs\varepsilon_{yy} + \left(c^2 - s^2\right)\varepsilon_{xy}$$

The transformation for the three strain components can be written in matrix form as

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & \left(c^2 - s^2\right) \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix}$$

10.2 Transformation of stress – 2D

The triangle shown in Figure 6 has two of its faces orthogonal to the x and y axis respectively. The remaining face, whose area is equal to α, is orthogonal to the unitary vector u.

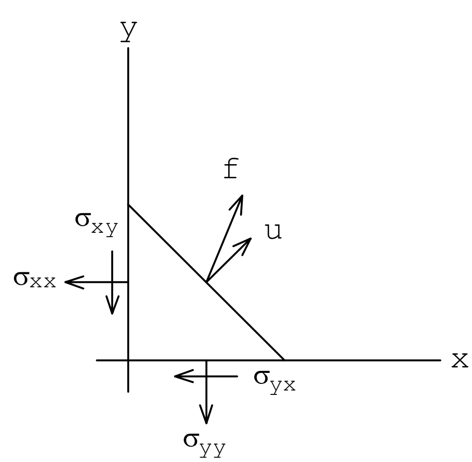

Figure 6

The projection of the area α on the plane orthogonal to the x and y axis can be written respectively as

$$\alpha_x = \alpha u_x$$

$$\alpha_y = \alpha u_y$$

The equilibrium of forces acting on the faces of the triangle can be written as

$$f = \begin{bmatrix} \sigma_{xx}\alpha_x + \sigma_{yx}\alpha_y \\ \sigma_{xy}\alpha_x + \sigma_{yy}\alpha_y \end{bmatrix} = \alpha \begin{bmatrix} \sigma_{xx}u_x + \sigma_{yx}u_y \\ \sigma_{xy}u_x + \sigma_{yy}u_y \end{bmatrix}$$

The stress vector acting on the face orthogonal to the vector u can be written as

$$\frac{f}{\alpha} = \begin{bmatrix} \sigma_{xx}u_x + \sigma_{yx}u_y \\ \sigma_{xy}u_x + \sigma_{yy}u_y \end{bmatrix}$$

$$\sigma_{yx} = \sigma_{xy} \Rightarrow \frac{f}{\alpha} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

The component of the stress vector, acting on a plane orthogonal to vector u^1, in the direction of a vector u^2, can be writen as

$$\sigma = \begin{bmatrix} u_x^2 & u_y^2 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} u_x^1 \\ u_y^1 \end{bmatrix}$$

$$\sigma = \sigma_{xx}u_x^1u_x^2 + \sigma_{yy}u_y^1u_y^2 + \sigma_{xy}\left(u_x^1u_y^2 + u_y^1u_x^2\right)$$

Consider Figure 5 that shows a reference system xy which has been rotated by an angle θ from the reference system xy.

$$u^1 = \begin{bmatrix} +c \\ +s \end{bmatrix}, \quad u^2 = \begin{bmatrix} -s \\ +c \end{bmatrix}$$

where,

$$c = \cos\theta, \quad s = \sin\theta$$

The stress xx can be written as follows:

$$\sigma_{xx} = c^2\sigma_{xx} + s^2\sigma_{yy} + 2cs\sigma_{xy}$$

The stress yy can be written as follows:

$$\sigma_{yy} = s^2\sigma_{xx} + c^2\sigma_{yy} - 2cs\sigma_{xy}$$

The stress xy can be written as follows:

$$\sigma_{xy} = -cs\sigma_{xx} + cs\sigma_{yy} + \left(c^2 - s^2\right)\sigma_{xy}$$

The transformation for the three stress components can be written in matrix form as

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & \left(c^2 - s^2\right) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

11 References

[ANSYS 2006] ANSYS Release 10.0, 2006.

[Barnes 1999] Barnes, M., 1999. Form finding and analysis of tension structures by dynamic relaxation. International Journal of Space Structures, 14, (2), 89-105.

[Coyette and Guisset 1988] Coyette, J. P. and Guisset, P., 1988. Cable network analysis by a nonlinear programming technique. Engineering Structures, 10, 41-46.

[Gill and Murray 1974] Gill, P. E. and Murray, W., 1974. Newton type methods for unconstrained and linearly constrained optimization. Mathematical Programming 7, 311-350.

[Nocedal and Wright 1999] Nocedal, J. and Wright, S. J., 1999. Numerical Optimization, Springer-Verlag.