The incremental Newton Raphson method

A motivation that leads naturally to the definition of Newton Raphson method will be shown first. Then, the incremental strategy will be introduced in an attempt to eliminate a flaw found in the NR method. The NR method together with the incremental strategy is called incremental NR method. It will be clear why this strategy is not always successful.

In the incremental NR method, the approach is the direct resolution of the system of nonlinear equations given by the equilibrium equation. It is important to note that the direct resolution of the equilibrium equation can result in items that represent positions of unstable equilibrium for the structure.

For the establishment of the NR method consider a system of n nonlinear equations with n unknowns, or the equilibrium equation in the context of structural analysis, given by:

$$r(x) = 0$$

The solution of this equation will be denoted by x^*. Each component of vector r is a nonlinear function given by:

$$r_i(x) : R^n \to R$$

An approximation for each function $r_i(x)$ can be obtained by the Taylor series expansion around any point x^0:

$$r_i(x) \approx r_i\left(x^0\right) + \sum_{j=1}^{n} \frac{\partial r_i}{\partial x_j}\left(x_j - x_j^0\right)$$

Consider a matrix S given by:

$$s_{ij} = \frac{\partial r_i}{\partial x_j}$$

An approximation for the equilibrium equation can be written as:

$$r(x) \approx r\left(x^0\right) + S\left(x^0\right)\left(x - x^0\right)$$

The matrix S(x) is known as the Jacobian matrix of r(x). In structural analysis, the points x represent the nodal displacements of the structure, the vector r(x) is the residue vector and S(x) represents the stiffness matrix.

Taking the second member of the previous equation as an acceptable approximation to r(x), then an approximate solution to r(x) = 0, can be written as:

$$r\left(x^0\right) + S\left(x^0\right)\left(x - x^0\right) = 0 \Rightarrow x = x^0 - S^{-1}\left(x^0\right) r\left(x^0\right)$$

The point x can be taken as a new point x^0. Continuing this way, a better approximation can be obtained for x^*. The convergence of this process can be demonstrated taking up a starting point sufficiently close to the solution. The flaw is that the starting point may not be sufficiently close to the solution.

1 Stopping criterion

In structural analysis, a usual stopping criterion is given by:

$$\left\| r\,(x) \right\|_\infty \le \varepsilon \left\| r\left(x^0\right) \right\|_\infty$$

This means that the maximum nodal unbalance is less than or equal to ε times the maximum nodal unbalance calculated for the structure at the configuration given by the starting point x^0. A usual value for ε is equal to 0.001.

It is important to add a counter to limit the number of equilibrium iterations because, depending on the value for ε, the stopping criterion can never be satisfied.

A constant concern with the accuracy of the results must exist. The use of double precision is recommended. A choice of appropriate units, so that the unknown displacements are as close as possible to unity, is also recommended.

2 The NR algorithm

In the following algorithm, the maximum number of equilibrium iterations is given by ne.

$e \leftarrow 0$
$x \leftarrow x^0$
while $(e < ne)$ and $\left[\|r(x)\|_\infty > \varepsilon \left\|r\left(x^0\right)\right\|_\infty\right]$ do
begin
$x \leftarrow x - S^{-1}(x)\, r(x)$
$e \leftarrow e + 1$
end

3 The incremental strategy

The incremental strategy consists in defining a parameterized problem by the value μ as:

$$r(x, \mu) = 0$$

The parameterized problem is defined such that a solution x^0 for $\mu = \mu_0$ is known and for $\mu = \mu_m$ it becomes the original problem.

$$r\left(x^0, \mu_0\right) = 0$$

$$r(x, \mu_m) = r(x)$$

The general idea is that a starting point for the solution of the problem defined by $\mu = \mu_s$ is the solution obtained in the previous problem defined by $\mu = \mu_{s-1}$.

Beginning with the solution for $\mu = \mu_0$, the solution for $\mu = \mu_m$ can be achieved by using the NR method in each successive value of the parameter μ.

4 The incremental NR algorithm

In the following algorithm, the number of increments or steps is given by ns.

$x \leftarrow x^0$

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for s = 1 to ns do
      begin
      e ← 0
      η ← ε ‖r (x, μs)‖∞
      while (e < ne) and [‖r (x, μs)‖∞ > η] do
            begin
            x ← x − S⁻¹ (x) r (x, μs)
            e ← e + 1
            end
      end
```

5 The equilibrium equation

In order to use the incremental NR method in structural analysis, it is appropriate to write the equilibrium equation in the following form:

$$r(x) = p(x) - f$$

In this equation, p(x) represents the internal forces generated by the structure in order to balance the external applied forces f. The parameterized problem can be defined as:

$$r(x, \mu) = p(x) - \mu f$$

Note that for $\mu = 0$, and considering all known displacements equal to zero, the undeformed position of the structure given by x = 0 is a solution of the previous equation. The original problem is obtained for $\mu = 1$.

The sequence of values for the parameter μ can be defined in terms of the step s as follows:

$$\mu_s = \frac{s}{ns} \mid s = 0, 1, \ldots, ns$$

The case where known displacements are not equal to zero can be treated by applying the incremental strategy to these displacements.

As a general rule, the number of steps (ns) should be chosen such that the NR method requires about 5 equilibrium iterations for convergence in each step. However, note that there is no guarantee that the incremental strategy will produce starting points sufficiently close to the solutions.

6 The incremental NR algorithm for structural analysis

In the following algorithm, the maximum number of equilibrium iterations is given by ne and the number of steps is given by ns.

$x \leftarrow 0$
for s = 1 to ns do
begin
$e \leftarrow 0$
$\mu \leftarrow \frac{s}{ns}$
$\eta \leftarrow \varepsilon \left\| p(x) - \mu f \right\|_\infty$
while $(e < ne)$ and $\left[\left\| p(x) - \mu f \right\|_\infty > \eta \right]$ do
begin
$x \leftarrow x - S^{-1}(x) \left[p(x) - \mu f \right]$
$e \leftarrow e + 1$
end
end

7 The decremental Newton Raphson method

It is important to note that it is impossible to start the incremental NR method with any other point than x = 0. However, this difficulty can be removed by defining the parameterized problem as:

$$r(x, \mu) = r(x) - \mu r\left(x^0\right)$$

Note that for $\mu = 1$, x^0 is a solution of this equation. The original problem is obtained for $\mu = 0$.

In the context of structural analysis, this definition of the parameterized problem means adding a fictitious force in the structure, given by $-r(x^0)$, in order to result equilibrium with the chosen point to start the incremental

NR method. The method proceeds by decreasing this force until its complete elimination.