A SIMPLE PROCEDURE FOR SHAPE FINDING AND ANALYSIS OF FABRIC STRUCTURES

Prof. Dr. Vinicius F. Arcaro, UNICAMP/FEC

Abstract

This text presents a mathematical modeling of a membrane finite element. It includes a total Lagrangian description using the Green strain definition and assumes a linear elastic material. A procedure to define the shape of a fabric structure and to analyze it in the presence of conservative forces and small strains is summarized. The shapes are generated by loading a membrane with concentrated forces, distributed force and also by prescribing displacements. Mathematical programming techniques make the use of stiffness matrix pointless.

Notation

The following applies unless otherwise specified or made clear by the context. A Greek letter expresses a scalar. A vector is always a column matrix and a lower case letter expresses it. An upper case letter expresses a matrix.

Finite element definition

Figure 1 shows a reference system with the xy plane located in the plane of the element. The nodes are labeled 1, 2 and 3 while traversing the sides in counter-clockwise fashion. The respective internal angles are labeled α_1, α_2 and α_3. The side is labeled with the number of its opposite node. The strains are assumed constant over the element and the material homogeneous and isotropic.

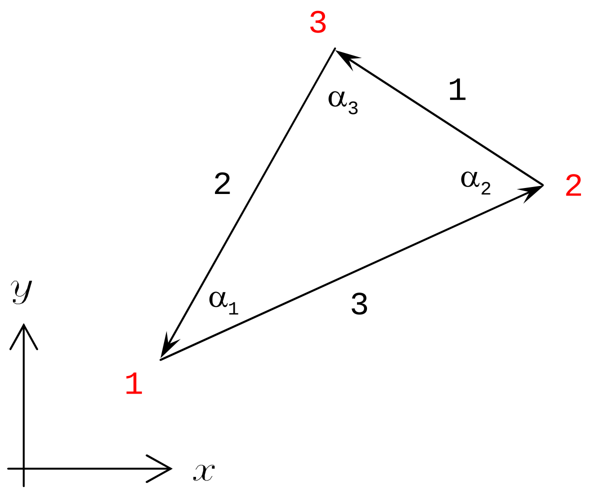

Figure 1

Directional strain

Considering the Green strain definition, the strain of an infinitesimal line segment in the direction of a unitary vector u, for a plane strain field, can be written as:

$$\varepsilon = c^2\varepsilon_{xx} + s^2\varepsilon_{yy} + 2cs\varepsilon_{xy}$$

Where,

$$u = \begin{bmatrix} c \\ s \end{bmatrix}$$

The directional strains for the directions of the sides of the triangle leads to the following:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} c_1^2 & s_1^2 & \sqrt{2}c_1s_1 \\ c_2^2 & s_2^2 & \sqrt{2}c_2s_2 \\ c_3^2 & s_3^2 & \sqrt{2}c_3s_3 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \sqrt{2}\varepsilon_{xy} \end{bmatrix}$$

$$\varepsilon = C\bar{\varepsilon}$$

Where,

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \quad , \quad C = \begin{bmatrix} c_1^2 & s_1^2 & \sqrt{2}c_1s_1 \\ c_2^2 & s_2^2 & \sqrt{2}c_2s_2 \\ c_3^2 & s_3^2 & \sqrt{2}c_3s_3 \end{bmatrix} \quad , \quad \bar{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \sqrt{2}\varepsilon_{xy} \end{bmatrix}$$

It is easy to show that,

$$CC^T = A = \begin{bmatrix} 1 & \cos^2\alpha_3 & \cos^2\alpha_2 \\ \cos^2\alpha_3 & 1 & \cos^2\alpha_1 \\ \cos^2\alpha_2 & \cos^2\alpha_1 & 1 \end{bmatrix}$$

Strain energy density

The strain energy density for a linearly elastic body can be written as:

$$\varphi = \frac{1}{2}\left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + 2\sigma_{xy}\varepsilon_{xy}\right)$$

Considering the previous definition of vector $\bar{\varepsilon}$, the strain energy density can be written as:

$$\varphi = \frac{1}{2}\bar{\varepsilon}^T\bar{\sigma}$$

Where,

$$\bar{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sqrt{2}\sigma_{xy} \end{bmatrix}$$

Constitutive relationship

A linear stress strain relationship is assumed according to the following expression:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{\left(1-\nu^2\right)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu) \end{bmatrix}\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix}$$

E is the Young’s modulus and ν is the Poisson’s ratio. Considering the previous definitions of vectors $\bar{\sigma}$ and $\bar{\varepsilon}$, the linear stress strain relationship can be written as:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sqrt{2}\sigma_{xy} \end{bmatrix} = \frac{E}{\left(1 - \nu^2\right)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1 - \nu) \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \sqrt{2}\varepsilon_{xy} \end{bmatrix}$$

$$\overline{\sigma} = \overline{H}\overline{\varepsilon}$$

Where,

$$\overline{H} = \frac{E}{\left(1 - \nu^2\right)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1 - \nu) \end{bmatrix}$$

Potential strain energy

The strain energy density can be written as:

$$\varphi = \frac{1}{2} \overline{\varepsilon}^T \overline{\sigma} = \frac{1}{2} \varepsilon^T C^{-T} \overline{H} C^{-1} \varepsilon$$

$$H = C^{-T} \overline{H} C^{-1} \Rightarrow \varphi = \frac{1}{2} \varepsilon^T H \varepsilon = \varphi\left(\varepsilon_1, \varepsilon_2, \varepsilon_3\right)$$

The potential strain energy and its gradient, known as the internal forces vector, can be written as:

$$\phi = \int_V \varphi\left(\varepsilon_1, \varepsilon_2, \varepsilon_3\right) dv \Rightarrow \frac{\partial \phi}{\partial x_i} = \int_V \left(\frac{\partial \varphi}{\partial \varepsilon_1} \frac{\partial \varepsilon_1}{\partial x_i} + \frac{\partial \varphi}{\partial \varepsilon_2} \frac{\partial \varepsilon_2}{\partial x_i} + \frac{\partial \varphi}{\partial \varepsilon_3} \frac{\partial \varepsilon_3}{\partial x_i} \right) dv$$

$$\sigma = H\varepsilon \Rightarrow \phi = \frac{1}{2} \int_V \varepsilon^T \sigma dv \Rightarrow \frac{\partial \phi}{\partial x_i} = \int_V \sigma^T \frac{\partial \varepsilon}{\partial x_i} dv$$

It is essential to note that the expressions of potential strain energy and its gradient can be written from any reference system - the xy plane does not need to be located in the plane of the element. It is easy to show that,

$$H = \frac{E}{\left(1 - \nu^2\right)} A^{-1} \left(I + \nu B A^{-1}\right)$$

Where,

$$B = \begin{bmatrix} 0 & \sin^2 \alpha_3 & \sin^2 \alpha_2 \\ \sin^2 \alpha_3 & 0 & \sin^2 \alpha_1 \\ \sin^2 \alpha_2 & \sin^2 \alpha_1 & 0 \end{bmatrix}$$

Strain components and its derivatives

The nodal displacements vectors are numbered according to its node numbers as shown in Figure 2.

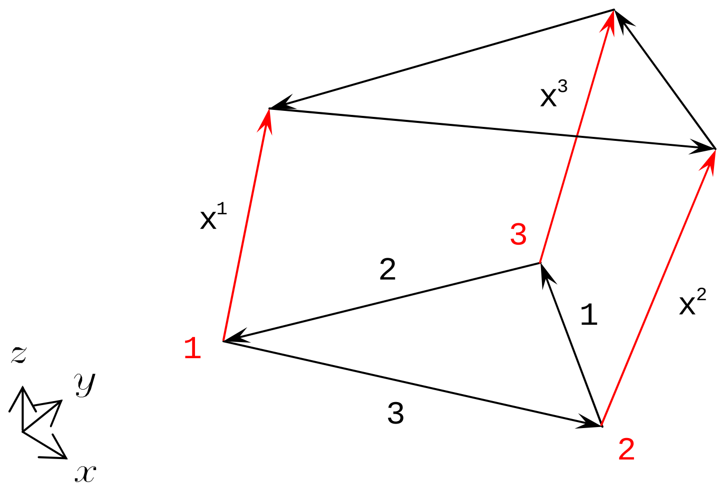

Figure 2

The nodal displacements are numbered according to:

$$x^1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad , \quad x^2 = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} \quad , \quad x^3 = \begin{bmatrix} x_7 \\ x_8 \\ x_9 \end{bmatrix}$$

To write the directional strain for a side of the triangle consider Figure 3, where u is a unitary vector parallel to the undeformed side, λ is the undeformed length of the side and p and q are the nodal displacements vectors.

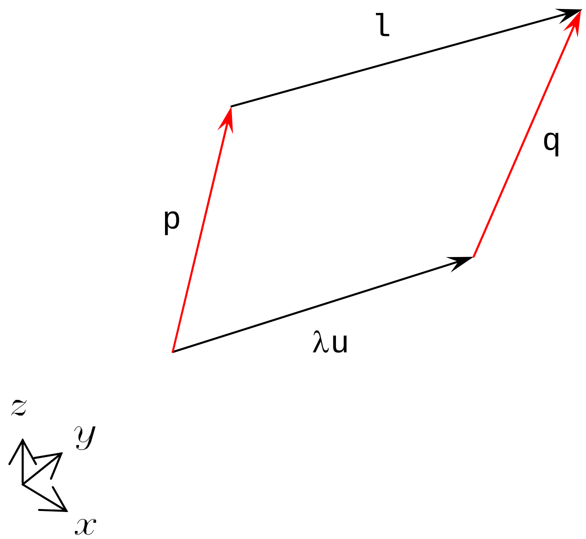

Figure 3

$$\lambda u + q - l - p = 0$$

$$l = \lambda u + q - p$$

$$z = \frac{q - p}{\lambda}$$

$$l = \lambda \left(u + z\right)$$

$$\|l\|^2 = \lambda^2 \left(1 + 2u^T z + z^T z\right)$$

$$\varepsilon = \frac{\|l\|^2 - \lambda^2}{2\lambda^2} = \frac{2u^T z + z^T z}{2}$$

$$\frac{\partial \varepsilon}{\partial p_i} = -\frac{1}{\lambda}\left(u_i + z_i\right)$$

$$\frac{\partial \varepsilon}{\partial q_i} = +\frac{1}{\lambda}\left(u_i + z_i\right)$$

Considering u^k as unitary vector parallel to the undeformed side k and λ_k as undeformed length of side k, the strain components and its derivatives can be written as:

$$z^1 = \frac{x^3 - x^2}{\lambda_1} \quad , \quad \varepsilon_1 = \frac{2\left(u^1\right)^T z^1 + \left(z^1\right)^T z^1}{2} \quad , \quad \nabla\varepsilon_1 = \frac{1}{\lambda_1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ -\left(u_1^1 + z_1^1\right) \\ -\left(u_2^1 + z_2^1\right) \\ -\left(u_3^1 + z_3^1\right) \\ +\left(u_1^1 + z_1^1\right) \\ +\left(u_2^1 + z_2^1\right) \\ +\left(u_3^1 + z_3^1\right) \end{bmatrix}$$

$$z^2 = \frac{x^1 - x^3}{\lambda_2} \quad , \quad \varepsilon_2 = \frac{2\left(u^2\right)^T z^2 + \left(z^2\right)^T z^2}{2} \quad , \quad \nabla\varepsilon_2 = \frac{1}{\lambda_2}\begin{bmatrix} +\left(u_1^2 + z_1^2\right) \\ +\left(u_2^2 + z_2^2\right) \\ +\left(u_3^2 + z_3^2\right) \\ 0 \\ 0 \\ 0 \\ -\left(u_1^2 + z_1^2\right) \\ -\left(u_2^2 + z_2^2\right) \\ -\left(u_3^2 + z_3^2\right) \end{bmatrix}$$

$$z^3 = \frac{x^2 - x^1}{\lambda_3} \quad , \quad \varepsilon_3 = \frac{2\left(u^3\right)^T z^3 + \left(z^3\right)^T z^3}{2} \quad , \quad \nabla\varepsilon_3 = \frac{1}{\lambda_3}\begin{bmatrix} -\left(u_1^3 + z_1^3\right) \\ -\left(u_2^3 + z_2^3\right) \\ -\left(u_3^3 + z_3^3\right) \\ +\left(u_1^3 + z_1^3\right) \\ +\left(u_2^3 + z_2^3\right) \\ +\left(u_3^3 + z_3^3\right) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Equilibrium configurations

The stable equilibrium configurations correspond to local minimum points of the total potential energy function. It is advisable the use of a Quasi Newton type method to find these local minimums because it does not requires the evaluation of the stiffness matrix.

Considering x as the vector of unknown displacements and f as the vector of nodal forces, the total potential energy function and its gradient can be written as:

$$\pi(x) = \sum_{\text{elements}} \phi(x) - f^T x$$

$$\nabla\pi(x) = \sum_{\text{elements}} \nabla\phi(x) - f$$

Principal stresses

To write the principal stresses for an element consider Figure 4, which shows a reference system with the xy plane located in the plane of the element.

$$\sigma = H\varepsilon$$

$$H = C^{-T}\overline{H}C^{-1} \Rightarrow \sigma = C^{-T}\overline{H}C^{-1}\varepsilon$$

$$\varepsilon = C\overline{\varepsilon} \Rightarrow \sigma = C^{-T}\overline{H}\overline{\varepsilon}$$

$$\overline{\sigma} = \overline{H}\overline{\varepsilon} \Rightarrow \sigma = C^{-T}\overline{\sigma} \Rightarrow \overline{\sigma} = C^{T}\sigma$$

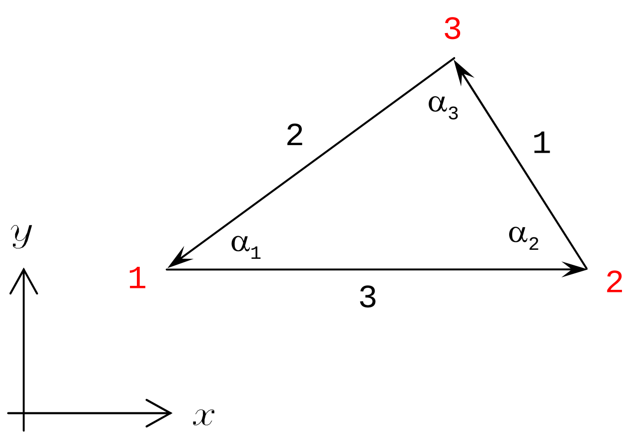

Figure 4

$$C = \begin{bmatrix} \cos^2\alpha_2 & \sin^2\alpha_2 & -\sqrt{2}\cos\alpha_2\sin\alpha_2 \\ \cos^2\alpha_1 & \sin^2\alpha_1 & +\sqrt{2}\cos\alpha_1\sin\alpha_1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\bar{\sigma} = C^T\sigma \Rightarrow \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \cos^2\alpha_2 & \cos^2\alpha_1 & 1 \\ \sin^2\alpha_2 & \sin^2\alpha_1 & 0 \\ -\cos\alpha_2\sin\alpha_2 & \cos\alpha_1\sin\alpha_1 & 0 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}$$

$$\Delta = \left(\sigma_{xx} - \sigma_{yy}\right)^2 + 4\sigma_{xy}^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 +$$
$$4\sigma_1\sigma_2\left(\cos^2\alpha_3 - \frac{1}{2}\right) + 4\sigma_1\sigma_3\left(\cos^2\alpha_2 - \frac{1}{2}\right) + 4\sigma_2\sigma_3\left(\cos^2\alpha_1 - \frac{1}{2}\right)$$

$$\sigma' = \frac{\left(\sigma_{xx} + \sigma_{yy}\right) \pm \sqrt{\Delta}}{2} = \frac{\left(\sigma_1 + \sigma_2 + \sigma_3\right) \pm \sqrt{\Delta}}{2}$$

Example

A procedure to define the shape of a fabric structure and to analyze it in the presence of conservative forces and small strains is summarized. Note that when strains are small, the Green strain is a reasonable approximation to the Engineering strain.

Step 1: The shape is generated by prescribing displacements to the initially plane membrane shown in Figure 5.

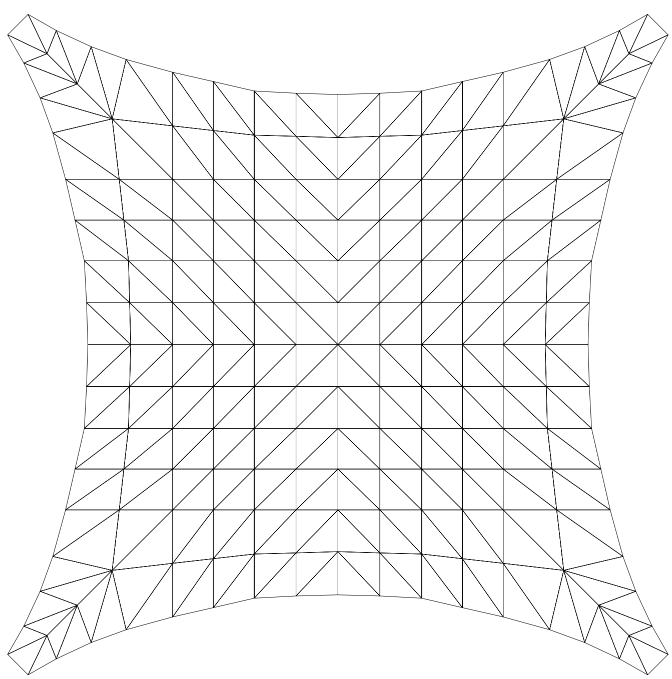

Figure 5

Two opposite corners are prescribed a positive displacement in the z-axis, which is perpendicular to the plane that contains the membrane, while the other two opposite corners are prescribed a negative displacement in the z-axis. The resulting shape is shown in Figure 6.

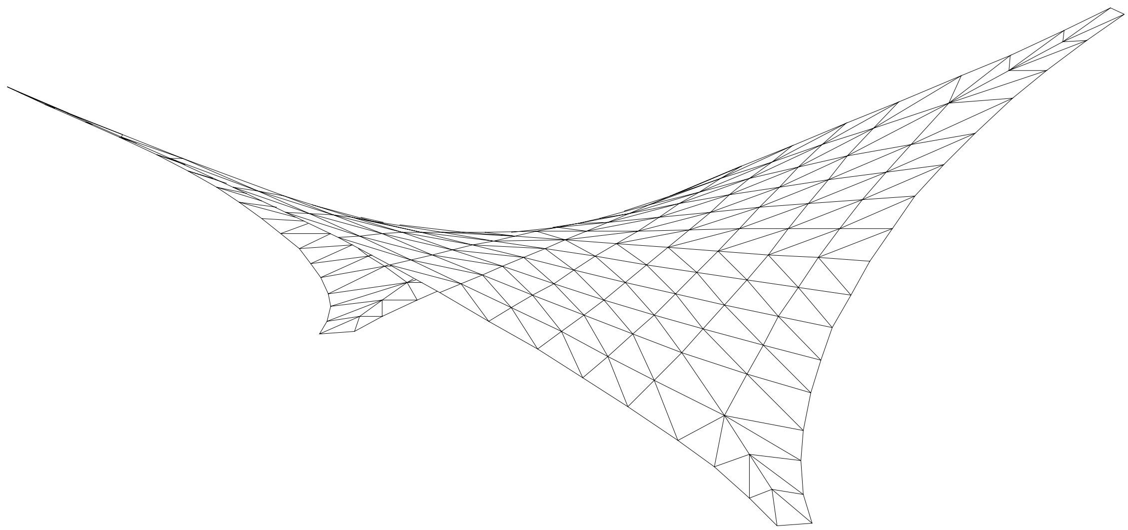

Figure 6

Step-2: The shape defined in the previous step is now used as the undeformed shape of the structure. Note that using an undeformed shape to analyze a fabric structure implies that patterns to build it do not need to compensate for strains in the membrane.

A single loading acting upward, similar to weight in nature, is applied as a crude simulation to wind uplift action. The result is shown in Figure 7.

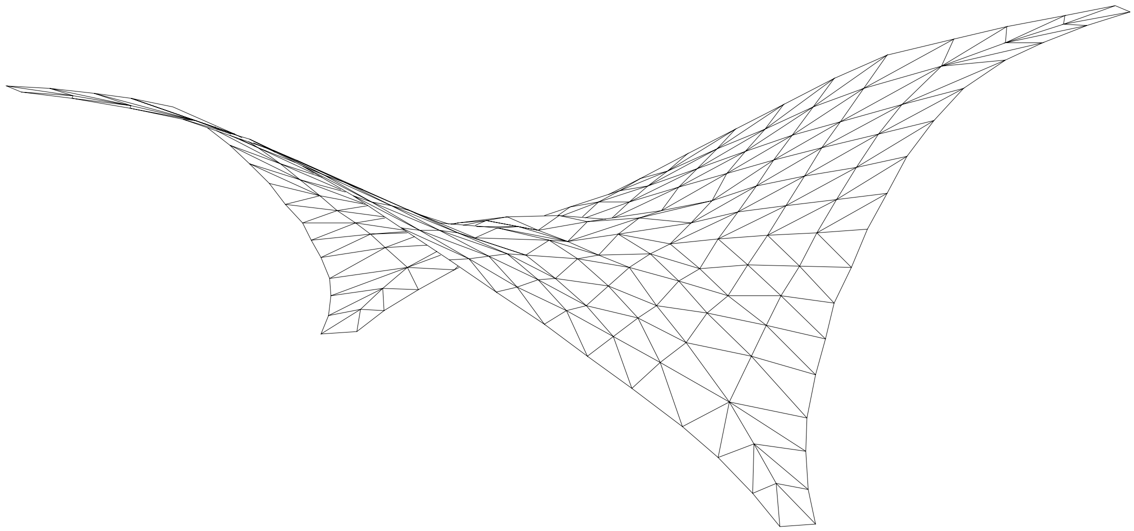

Figure 7

As can be seen in Figure 7, parts of the structure are flaccid. This flaccidity is due the fact that the upward loading tends to increase tension in one part of the structure and decrease tension in another part of the structure. Since, the structure was undeformed, this is no surprise - the structure needs to be tensioned. The tensioning must be determined such that the upward loading produces no flaccidity. Prescribing displacements to the undeformed structure as shown in Figure 8 (red segments) may result in the required tensioning.

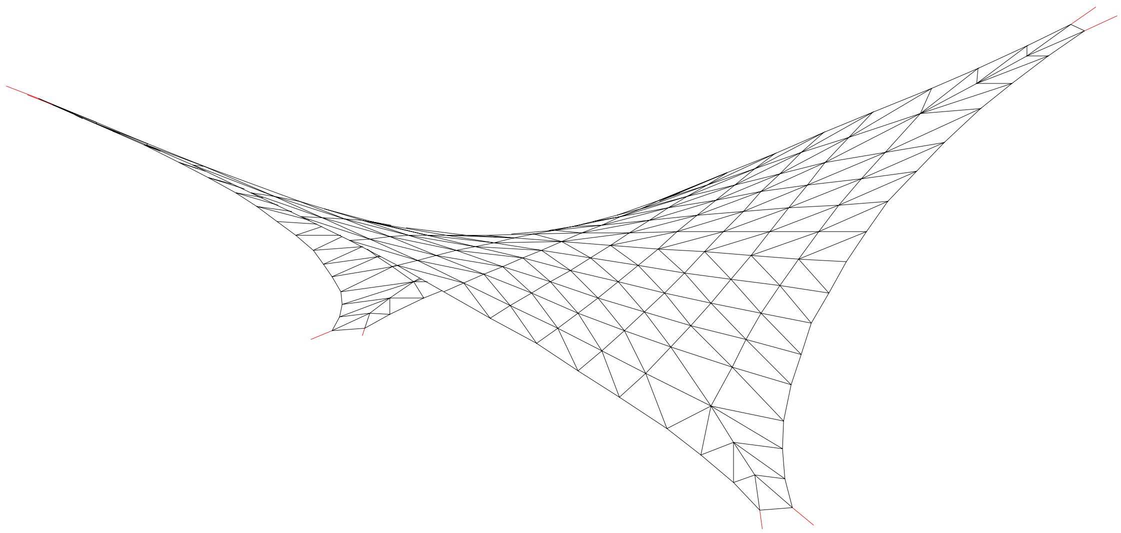

Figure 8

Applying both the prescribed displacements and the loading acting upward results in a deformed shape shown in Figure 9. In practice the tensioning process, which is simply another loading to the structure, would be achieved through tensioning steel cables passing on the edge of the membrane.

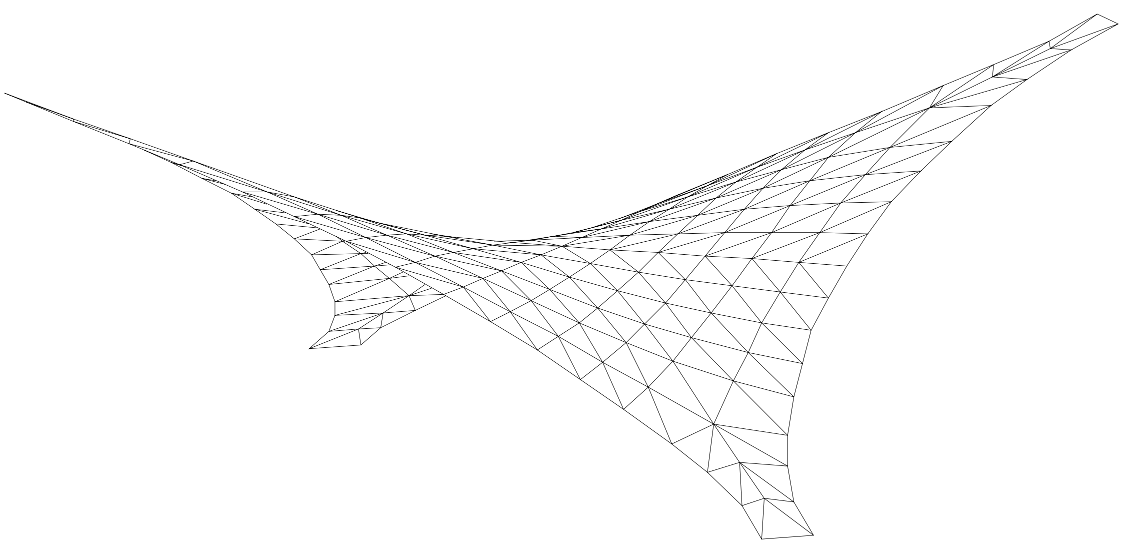

Figure 9

Computational performance

Table 1 shows the computational performance on an ordinary Pentium machine (200 MHz). The Limited Memory BFGS method was used. The line search procedure used cubic interpolation.

Table 1

	Shape Finding	Analysis	
		Without Tensioning	With Tensioning
Iterations	65	945	168
CPU time (s)	3	33	6

Bibliography

- Gill, P. E. and Murray, W., Newton type methods for unconstrained and linearly constrained optimization, Mathematical Programming 7, 1974.
- Lasdon, L. S., Optimization theory for large systems, Macmillan, New York, 1970.
- Luenberger, D. G., Linear and nonlinear programming, second edition, Addison Wesley, Reading, Massachusetts, 1989.
- Nocedal, J. and Wright, S. J., Numerical Optimization, Springer-Verlag, 1999.

APPENDIX

Trigonometry

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Geometry

Figure 10 shows a reference system with the xy plane located in the plane of the element. A positive angle θ from the x axis can be used to define the direction of the side of the triangle.

$c_i = \cos\theta_i$, $s_i = \sin\theta_i$, $i = 1,2,3$

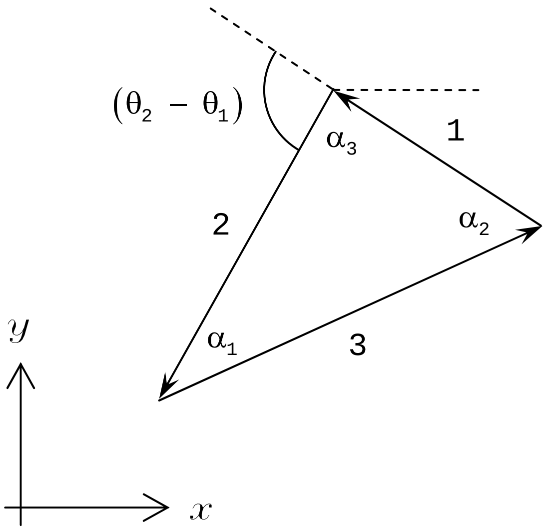

Figure 10

$(\theta_2 - \theta_1) + \alpha_3 = \pi \Rightarrow \cos(\theta_2 - \theta_1) = -\cos\alpha_3$

$(\theta_2 - \theta_1) + \alpha_3 = \pi \Rightarrow \sin(\theta_2 - \theta_1) = -\sin\alpha_3$

The matrix A

$$A = CC^T = \begin{bmatrix} c_1^2 & s_1^2 & \sqrt{2}c_1s_1 \\ c_2^2 & s_2^2 & \sqrt{2}c_2s_2 \\ c_3^2 & s_3^2 & \sqrt{2}c_3s_3 \end{bmatrix} \begin{bmatrix} c_1^2 & c_2^2 & c_3^2 \\ s_1^2 & s_2^2 & s_3^2 \\ \sqrt{2}c_1s_1 & \sqrt{2}c_2s_2 & \sqrt{2}c_3s_3 \end{bmatrix}$$

$a_{11} = c_1^4 + s_1^4 + 2c_1^2s_1^2 = \left(c_1^2 + s_1^2\right)^2 = 1$

In a similar way,

$a_{22} = 1$, $a_{33} = 1$

$a_{21} = c_1^2c_2^2 + s_1^2s_2^2 + 2c_1s_1c_2s_2 = \left(c_1c_2 + s_1s_2\right)^2$

$\left(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2\right)^2 = \cos^2(\theta_2 - \theta_1) = \cos^2\alpha_3$

$a_{21} = \cos^2\alpha_3$

In a similar way,

$$a_{31} = \cos^2\alpha_2 \;,\; a_{32} = \cos^2\alpha_1$$

$$A = \begin{bmatrix} 1 & \cos^2\alpha_3 & \cos^2\alpha_2 \\ \cos^2\alpha_3 & 1 & \cos^2\alpha_1 \\ \cos^2\alpha_2 & \cos^2\alpha_1 & 1 \end{bmatrix}$$

The matrix H

$$\bar{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \bar{H} = \frac{E}{\left(1 - \nu^2\right)}\left(I + \nu\bar{B}\right)$$

$$H = C^{-T}\bar{H}C^{-1}$$

$$A = CC^T \Rightarrow C^{-T} = A^{-1}C \Rightarrow H = A^{-1}C\bar{H}C^{-1}$$

$$\bar{H} = \frac{E}{\left(1 - \nu^2\right)}\left(I + \nu\bar{B}\right) \Rightarrow H = \frac{E}{\left(1 - \nu^2\right)} A^{-1}\left(I + \nu C\bar{B}C^{-1}\right)$$

$$A = CC^T \Rightarrow C^{-1} = C^TA^{-1} \Rightarrow H = \frac{E}{\left(1 - \nu^2\right)} A^{-1}\left[I + \nu\left(C\bar{B}C^T\right)A^{-1}\right]$$

$$B = C\bar{B}C^T = \begin{bmatrix} c_1^2 & s_1^2 & \sqrt{2}c_1s_1 \\ c_2^2 & s_2^2 & \sqrt{2}c_2s_2 \\ c_3^2 & s_3^2 & \sqrt{2}c_3s_3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} c_1^2 & c_2^2 & c_3^2 \\ s_1^2 & s_2^2 & s_3^2 \\ \sqrt{2}c_1s_1 & \sqrt{2}c_2s_2 & \sqrt{2}c_3s_3 \end{bmatrix}$$

$$b_{11} = c_1^2s_1^2 + c_1^2s_1^2 - 2c_1s_1c_1s_1 = 0$$

In a similar way,

$$b_{22} = 0 \;,\; b_{33} = 0$$

$$b_{21} = c_2^2s_1^2 + c_1^2s_2^2 - 2c_2s_2c_1s_1 = \left(s_2c_1 - s_1c_2\right)^2$$

$$\sin^2\left(\theta_2 - \theta_1\right) = \left(\sin\theta_2\cos\theta_1 - \sin\theta_1\cos\theta_2\right)^2 = \sin^2\alpha_3$$

$$b_{21} = \sin^2\alpha_3$$

In a similar way,

$$b_{31} = \sin^2 \alpha_2 \ , \ b_{32} = \sin^2 \alpha_1$$

$$B = \begin{bmatrix} 0 & \sin^2 \alpha_3 & \sin^2 \alpha_2 \\ \sin^2 \alpha_3 & 0 & \sin^2 \alpha_1 \\ \sin^2 \alpha_2 & \sin^2 \alpha_1 & 0 \end{bmatrix}$$

$$H = \frac{E}{\left(1 - \nu^2\right)} A^{-1} \left(I + \nu B A^{-1}\right)$$