A SIMPLE PROCEDURE FOR ANALYSIS OF CABLE NETWORK STRUCTURES

Prof. Dr. Vinicius F. Arcaro, UNICAMP/FEC

Abstract
This text presents a mathematical modeling of a cable finite element. It includes a total Lagrangian description using the Engineering strain definition and assumes an elastic material (linear or nonlinear). A procedure to analyze a cable network in the presence of conservative forces and small deformations is summarized. Mathematical programming techniques make the use of stiffness matrix pointless.

Notation
The following applies unless otherwise specified or made clear by the context. A Greek letter expresses a scalar. A vector is always a column matrix and a lower case letter expresses it. An upper case letter expresses a matrix.

Finite element definition
The geometry of a one-dimensional cable element is shown in Figure 1. The nodes are labeled 1 and 2. The strain is assumed constant along the element and the material homogeneous and isotropic.

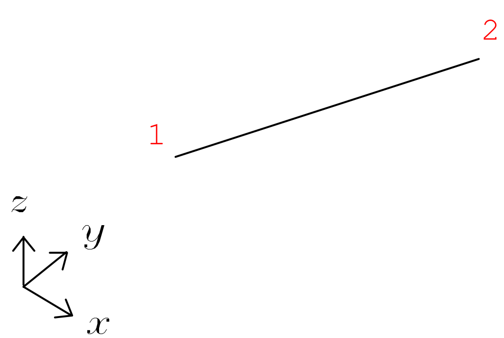

Figure 1

Deformed length
In Figure 2, the λu vector, where u is a unity vector, represents the cable element in a configuration with zero nodal displacements. It is easy to understand that λ represents the distance between the nodes of the element in this configuration. However, as will be explained latter, this distance will not always represent the undeformed

length of the element. The vector l represents the element in its deformed configuration. The vectors p and q represent the nodal displacements vectors.

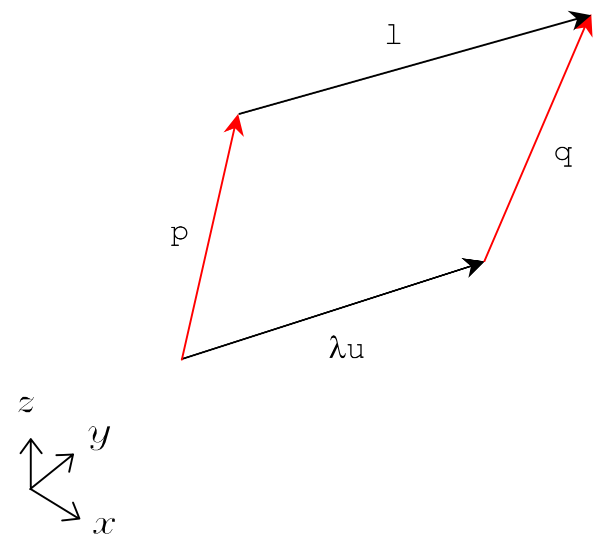

Figure 2

The deformed length can be found as follows:

$$\lambda u + q - l - p = 0$$

$$l = \lambda u + q - p$$

$$z = \frac{q - p}{\lambda} \Rightarrow l = \lambda\left(u + z\right)$$

$$\delta = 2u^T z + z^T z \Rightarrow \|l\| = \lambda\sqrt{1 + \delta}$$

Imposing a constant cut

Consider an element with undeformed length less than the *initial distance* of its nodes, where initial distance is defined as the distance of the two nodes with zero nodal displacements. This element can be pictured with a *cut* in its undeformed length. The result is that this element will show tension in any rigid body motion that preserves the initial distance of its nodes.

The Initial configuration of a cable network is defined as the configuration of zero nodal displacements for all its nodes. Applying imaginary cuts to selected elements of a cable network in its initial configuration is an easy way to apply tension to this cable network. Notice that if no cuts are present, the initial configuration is also the undeformed configuration.

It is worth mentioning that effects due to temperature change also can be treated through an imaginary cut in the undeformed length of the element.

Strain

Considering μ as the value of the cut in the undeformed length of an element, the *cut length* of this element can be written as:

$$\lambda_\mu = \lambda - \mu$$

After applying a cut, consider a change Δt in temperature. The coefficient of thermal expansion is denoted by α_t. This sequence leads to the *strain-free length* as:

$$\lambda_0 = \lambda (1 - \rho) (1 + \varepsilon_t)$$

Where,

$$\rho = \frac{\mu}{\lambda} \quad , \quad \varepsilon_t = \alpha_t \Delta t$$

The strain can be written as:

$$\varepsilon = \frac{\|1\| - \lambda_0}{\lambda_0}$$

$$\varepsilon = \frac{\sqrt{1 + \delta} - (1 - \rho) (1 + \varepsilon_t)}{(1 - \rho) (1 + \varepsilon_t)}$$

In order to avoid severe cancellation, the strain expression should be evaluated as:

$$\varepsilon = \frac{\dfrac{\delta}{\sqrt{1 + \delta} + 1} + \rho + \rho\varepsilon_t - \varepsilon_t}{(1 - \rho) (1 + \varepsilon_t)}$$

Potential strain energy

Considering σ as the conjugate stress to the engineering strain ε and α as the undeformed area of the element, the potential strain energy and its gradient can be written as:

$$\phi = \alpha\lambda_0 \int_0^{\varepsilon} \sigma(\xi)\, d\xi$$

$$\frac{\partial\phi}{\partial p_i} = \alpha\lambda_0 \sigma(\varepsilon) \frac{\partial\varepsilon}{\partial p_i} = -\frac{\alpha\sigma(\varepsilon)(u_i + z_i)}{\sqrt{1 + \delta}}$$

$$\frac{\partial\phi}{\partial q_i} = \alpha\lambda_0 \sigma(\varepsilon) \frac{\partial\varepsilon}{\partial q_i} = +\frac{\alpha\sigma(\varepsilon)(u_i + z_i)}{\sqrt{1 + \delta}}$$

Geometric interpretation

Considering Figure 2, a unit vector v parallel to the element in its deformed configuration can be written as:

$$v = \frac{l}{\|l\|} = \frac{u + z}{\sqrt{1 + \delta}}$$

Using vector v, the gradient of the potential strain energy can be written as:

$$\frac{\partial\phi}{\partial p_i} = -\alpha\sigma(\varepsilon)\, v_i$$

$$\frac{\partial\phi}{\partial q_i} = +\alpha\sigma(\varepsilon)\, v_i$$

Figure 3 shows the geometric interpretation of the gradient of the potential strain energy as forces acting on nodes of the element. These forces are known as internal forces.

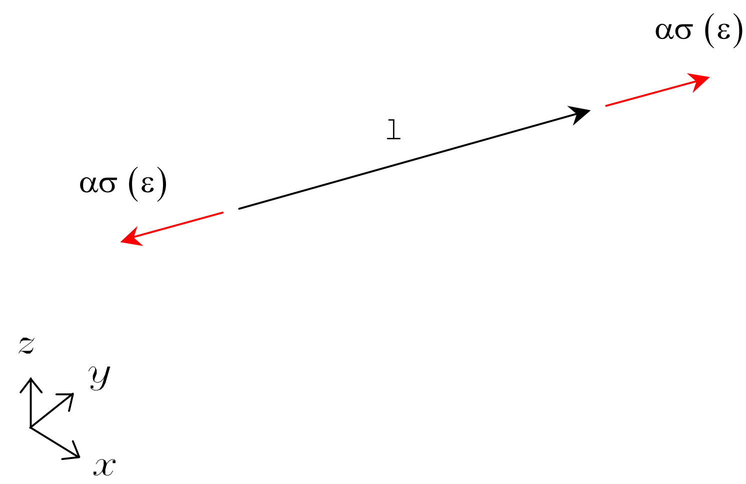

Figure 3

Imposing a constant tension

Consider the following scalar function ϕ and its gradient:

$$\phi = \alpha\sigma_0\lambda\sqrt{1 + \delta}$$

$$\frac{\partial\phi}{\partial p_i} = -\alpha\sigma_0 v_i$$

$$\frac{\partial\phi}{\partial q_i} = +\alpha\sigma_0 v_i$$

The gradient can be interpreted as internal forces, with constant modulus, acting on nodes of an element. The scalar function can be interpreted as the corresponding potential strain energy.

An element with imposed constant tension can be defined by choosing positive value for the constant stress σ_0. The result is that this element will show constant tension in any displacement of its nodes. Applying constant tension to selected elements of a cable network in its initial configuration is another easy way to apply tension to this cable network.

Equivalence between constant cut and constant tension

A constant cut value is equivalent to a constant tension value in the sense that they both produce the same internal forces. To find the equivalence between them, consider a cable network at a known configuration.

To find the constant cut value equivalent to the constant tension value, first find the strain ε according:

$$\sigma(\varepsilon) = \sigma_0$$

Then, find the cut value μ according:

$$\mu = \frac{\lambda}{(1+\varepsilon_t)(1+\varepsilon)}\left(\varepsilon + \varepsilon_t + \varepsilon_t\varepsilon - \frac{\delta}{1+\sqrt{1+\delta}}\right)$$

To find the constant tension value equivalent to the constant cut value, first find the strain ε according:

$$\varepsilon = \frac{\dfrac{\delta}{\sqrt{1+\delta}+1} + \rho + \rho\varepsilon_t - \varepsilon_t}{(1-\rho)(1+\varepsilon_t)}$$

Then, find the tension σ_0 according:

$$\sigma_0 = \sigma(\varepsilon)$$

Constitutive relationship
A linear stress strain relationship is assumed according to the following expression:

$$\sigma = E\varepsilon$$

Where E is the Young's modulus. The potential strain energy can be written as:

$$\phi = \alpha\lambda_0\int_0^{\varepsilon}\sigma(\xi)\,d\xi = \frac{1}{2}\alpha\lambda_0 E\varepsilon^2$$

Equilibrium configurations
The stable equilibrium configurations correspond to local minimum points of the total potential energy function. It is advisable the use of a Quasi Newton type method to find these local minimums because it does not requires the evaluation of the stiffness matrix.

Considering x as the vector of unknown displacements and f as the vector of nodal forces, the total potential energy function and its gradient can be written as:

$$\pi(x) = \sum_{\text{elements}} \phi(x) - f^T x$$

$$\nabla\pi(x) = \sum_{\text{elements}} \nabla\phi(x) - f$$

Example

A procedure to analyze a cable network in the presence of conservative forces and small deformations is summarized.

Figure 4 shows a cable network in its undeformed configuration. It is a cable beam, whose design is known as Zetlin.

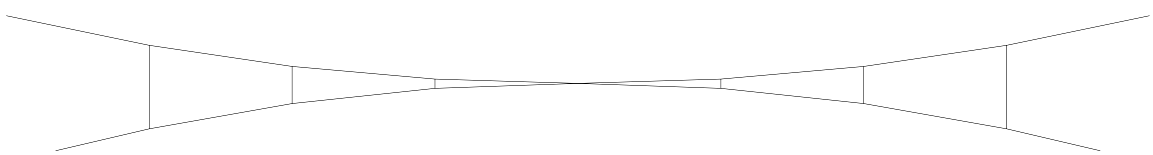

Figure 4

A loading consisting of forces acting upward is applied on nodes of the top cable as a crude simulation of wind uplift action. The wind action is considered as distributed load acting along the span of the top cable. The self-weight of cables is considered in this analysis.

This loading results in compression of the top cable elements in this model. This should be interpreted as the elements becoming slack or flaccid.

This flaccidity is due the fact that the upward loading tends to increase stress at the bottom cable and decrease stress at the top cable. Since the structure was undeformed, this result is no surprise - the structure needs to be tensioned. The tensioning must be determined such that the upward loading produces no flaccidity.

Figure 5 shows the cable network in its undeformed configuration with two elements marked in red.

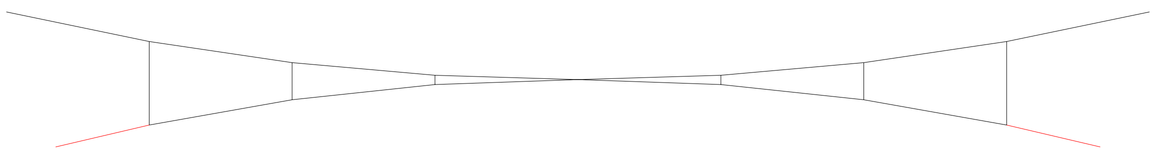

Figure 5

Imposing a constant tension to these elements may result in the required tensioning. Only the self-weight of cables is considered in this step.

It is important to notice that imposing a constant tension to selected elements of a cable network is an attempt to simulate what is accomplished through hydraulic jacks in practice.

The general problem is to choose a specific value for the constant tension that results in tension of all elements for all loading cases. A good trial value is to set the constant tension at a percentage of the breaking tension of the rope. The deformed structure can be said tensioned or pre-stressed.

Applying the constant cut value equivalent to the constant tension value found in the previous step, and the loading acting upward results in a deformed configuration shown in Figure 6, where all elements show tension.

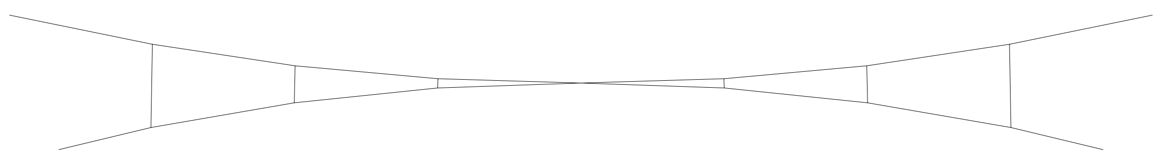

Figure 6

It is important to notice that imposing a constant cut equivalent to the constant tension is an attempt to simulate what happens after the hydraulic jacks have been removed. The action of a hydraulic jack is pictured as to shorten the selected element where it is applied.

Computational performance

Table 1 shows the computational performance on an ordinary Pentium machine (200 MHz). The Limited Memory BFGS method was used. The line search procedure used cubic interpolation.

	Loading	Tensioning	Tensioning Loading
Iterations	94	423	54
CPU time (s)	0	1	0

Bibliography

- Gill, P. E. and Murray, W., Newton type methods for unconstrained and linearly constrained optimization, Mathematical Programming 7, 1974.

- Lasdon, L. S., Optimization theory for large systems, Macmillan, New York, 1970.
- Luenberger, D. G., Linear and nonlinear programming, second edition, Addison Wesley, Reading, Massachusetts, 1989.
- Nocedal, J. and Wright, S. J., Numerical Optimization, Springer-Verlag, 1999.